

2 π -GRAFTINGS AND COMPLEX PROJECTIVE STRUCTURES I

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ABSTRACT. Let S be a closed orientable surface of genus at least two, and let C and C' be complex projective structures on S with the same holonomy and orientation. We show that, if, via Thurston's coordinates, the projection of C' to $\mathcal{PML}(S)$ is sufficiently close to that of C , then C and C' are related by a 2π -grafting along a multiloop M . Moreover M is well-approximated by the difference of the measured laminations corresponding to C and C' , calculated on a traintrack.

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1. INTRODUCTION

A (complex) projective structure on a connected and orientable surface F is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2, \mathbb{C}))$ -structure. That is, an atlas modeled on the Riemann sphere $\hat{\mathbb{C}}$ with transition maps lying in $\mathrm{PSL}(2, \mathbb{C})$. Equivalently (see for example [19]), a projective structure on F is a pair (f, ρ) of

- an immersion $f: \tilde{F} \rightarrow \hat{\mathbb{C}}$ (developing map), where \tilde{F} is the universal cover of F , and
- a representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ (holonomy)

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such that f is ρ -equivariant, i.e. $f \cdot \gamma = \rho(\gamma) \cdot f$ for all $\gamma \in \pi_1(F)$.

Throughout this paper, we consider only marked complex projective structures of fixed orientation. Our main focus is on projective structures on a closed orientable surface S of genus at least 2.

Then Gallo-Kapovich-Marden [8] showed that a representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is the holonomy representation of some projective structure (on S) if and only if ρ satisfies:

- $\mathrm{Im}(\rho)$ is nonelementary, and
- ρ lifts to $\tilde{\rho}: \pi_1(S) \rightarrow \mathrm{SL}(2, \mathbb{C})$.

In particular, holonomy representations are *not* necessarily discrete or faithful, in contrast with, for example, complete hyperbolic structures. Moreover, a representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ with the above conditions corresponds to infinitely many different projective structures (which is essentially in [8]; see also [3]).

Thus a basic question is the characterization of projective structures with fixed holonomy, which has indeed been raised in [10, 14, 8, 6]. However, answers to this question have been given only for some special discrete (and, in most cases, faithful) representations ([9, 11, 2]), using a certain surgery operation, called (2π) -grafting (see §3.1).

Let \mathcal{P}_ρ denote the set of projective structures on S with fixed holonomy $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ and fixed orientation, and let $\mathcal{PML}(S)$ denote the space of projective measured laminations on S , which is homeomorphic to \mathbb{S}^{6g-7} . Our goal of this paper is to give a local characterization of \mathcal{P}_ρ using grafting, where the locality is measured in $\mathcal{PML}(S)$ by a projection of \mathcal{P}_ρ into $\mathcal{PML}(S)$ (see Theorem A). This characterization is given for arbitrary holonomy ρ .

In the sequel paper [1], using this local characterization, we prove moreover that grafting indeed generates the entire set \mathcal{P}_ρ for every generic representation ρ in the $\mathrm{PSL}(2, \mathbb{C})$ -character variety $\chi(S)$ (see [8, Problem 12.1.2]).

1.1. Fuchsian projective structures. We review the characterization of \mathcal{P}_ρ when $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is a discrete and faithful representation onto $\mathrm{PSL}(2, \mathbb{R})$ (fuchsian representation). Accordingly $\mathrm{Im}(\rho) =: \Gamma$ is called a fuchsian group. Then quotienting out the domain of discontinuity of Γ by Γ , we obtain two projective structures on S with the fuchsian holonomy ρ , which have opposite orientations. Let C_0 denote the one in \mathcal{P}_ρ .

Theorem 1.1 (Goldman [9]; also [13]). *For every $C \in \mathcal{P}_\rho$, C can be obtained by grafting C_0 along a multiloop M on C .*

Then we can naturally regard the multiloop M as a measure lamination by assigning weight 2π to each loop of M .

Theorem 1.1 was proved, moreover, for quasifuchsian representations. Nevertheless, the proof easily boils down to the fuchsian case by a quasiconformal map.

1.2. Thurston's coordinates. (see §3.2) Let $\mathcal{P}(S)$ be the space of projective structures on S . W. Thurston gave a parametrization of $\mathcal{P}(S)$ in a geometric manner:

$$\mathcal{P}(S) \cong \text{Teich}(S) \times \mathcal{ML}(S),$$

where $\text{Teich}(S)$ is the Teichmüller space of S and $\mathcal{ML}(S)$ is the space of measured laminations on S . Namely, a pair $(\tau, L) \in \text{Teich}(S) \times \mathcal{ML}(S)$ represents a pleated surface in \mathbb{H}^3 and a projective structure $C \in \mathcal{P}(S)$ is modeled on $\hat{\mathbb{C}}$; recalling $\partial\mathbb{H}^3 \cong \hat{\mathbb{C}}$, a certain nearest-point projection from $\hat{\mathbb{C}}$ to the pleated surface yields the correspondence between C and (τ, L) .

In this paper, we mostly use Thurston's coordinates and write $C = (\tau, L)$, where $C \in \mathcal{P}(S)$, $\tau \in \text{Teich}(S)$ and $L \in \mathcal{ML}(S)$. We also write $C = (f, \rho)$ as a pair of the developing map f and the holonomy representation ρ of C . (We distinct them by notations.)

As an example of Thurston's coordinates, consider a projective structure $C = (\tau, L)$ with fuchsian holonomy $\rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$. Then τ is $\mathbb{H}^2/\text{Im}(\rho)$, and L is the multiloop M in Theorem 1.1 for C (see [9]).

1.3. Local characterization of \mathcal{P}_ρ in $\mathcal{PML}(S)$. The goal of this paper is to prove:

Theorem A (see Theorem 6.1). Let $C = (\tau, L)$ be a projective structure on S with arbitrary holonomy $\rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$. Then there exists a neighborhood U of $[L]$ in $\mathcal{PML}(S)$ such that, if $C' = (\tau', L') \in \mathcal{P}_\rho$ satisfies $[L'] \in U$, then C and C' are related by a grafting along a multiloop. More precisely, we have either

- (i) $C' = Gr_M(C)$ for some multiloop M that is a good approximation of the difference $L' - L$, which is calculated on some traintrack carrying both L and L' , or
- (ii) $[L] = [L'] \in \mathcal{PML}(S)$ and $C = Gr_M(C')$, where the multiloop $M = L - L'$.

We see that Theorem 1.1 is a special case of Theorem A (see Theorem 6.1 Case(III)): Let $\rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ be a fuchsian representation and let $C = C_0 \in \mathcal{P}_\rho$ as in §1.1. Then $C_0 = (\tau, \emptyset)$ in Thurston's

coordinates, where $\tau = \mathbb{H}^2 / \text{Im}(\rho) \in \text{Teich}(S)$. Then, in this special case, Theorem A holds for $C = C_0$ with $U = \mathcal{PML}(S)$ by Theorem 1.1. More precisely, for every $C' \in \mathcal{P}_\rho$, we have $C' = Gr_{L'}(C_0)$, where $C' = (\tau, L')$, and thus $M = L' - \emptyset = L'$. However, we do *not* use Theorem 1.1 to prove Theorem A.

A main step of the proof of Theorem A is to prove the following theorem, which well explains Theorem A:

Theorem B (see Proposition 5.15 and also Theorem 5.4). Let $C = (\tau, L)$ be a projective structure on S with holonomy ρ . For every neighborhood V of τ in $\text{Teich}(S)$, there exists a neighborhood U of $[L]$ in $\mathcal{PML}(S)$ such that, if a projective structure $C' = (\tau', L')$ in \mathcal{P}_ρ satisfies $[L'] \in U$, then $\tau' \in V$.

By Theorem B, if U is sufficiently small, then τ and τ' are very close in $\text{Teich}(S)$. Therefore, in Theorem A, the difference of the projective structures $C = (\tau, L)$ and $C' = (\tau', L')$ is captured, mostly, by the second coordinate $\mathcal{ML}(S)$ of Thurston's coordinates.

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1.4. Outline of the proofs. Roughly speaking, we use the following ideas.

1.4.1. Theorem B. It suffices to show that, for every $\epsilon > 0$, if the neighborhood U of $[L] \in \mathcal{PML}(S)$ is sufficiently small, then $\text{length}_{\tau'}(l') / \text{length}_\tau(l) < (1 + \epsilon)$ for all corresponding loops l and l' on τ and τ' , respectively (Proposition 5.16). Let $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ and $\beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending maps associated with the projective structures C and C' , respectively, where their domains \mathbb{H}^2 correspond to the universal covers of τ and τ' . Since β and β' are (almost everywhere) a locally isometry onto its image with intrinsic metric, we compare the lengths of l and l' by analyzing $\beta(\tilde{l})$ and $\beta'(\tilde{l}')$, where \tilde{l} and \tilde{l}' are the lifts of l and l' to \mathbb{H}^2 , respectively. Since β and β' are ρ -equivariant, they are ρ -equivariantly homotopic (Lemma 3.3). Moreover, since their bending laminations L and L' are sufficiently close, projected in $\mathcal{PML}(S)$, it turns out that β' is contained in a “small regular neighborhood” of β (§5.3), where β and β' are regarded as pleated surfaces in \mathbb{H}^3 . Then we have a ρ -equivariant “near point projection” from β' to β in \mathbb{H}^3 . Then this projection descends to a marking preserving map from τ_i to τ . Since β' is contained in a sufficiently small neighborhood of β , we can assume that this projection is $(1 + \epsilon)$ -lipschitz (Lemma 5.18). Therefore $\text{length}_{\tau'}(l') / \text{length}_\tau(l) < 1 + \epsilon$.

1.4.2. *Theorem A.* By Theorem B, we can identify τ and τ' by a smooth $(1 + \epsilon)$ -bilipschitz map, with sufficiently small $\epsilon > 0$, preserving the marking of the surface. Then, using this identification, we see that the bending maps β and β' are sufficiently close in C^0 -topology and, moreover, in C^1 -topology in the complement of a small neighborhood of the bending laminations corresponding to L and L' (Theorem 5.4). Then, recalling that $\tau, \tau' \in \text{Teich}(S)$, take a (fat) traintrack T on S that carries both L and L' . Let $\kappa: C \rightarrow \tau$ and $\kappa': C' \rightarrow \tau'$ be the *collapsing maps*, which realize the correspondence of Thurston's coordinates (§3.2.3). Then, since β and β' are C^1 -close in the complement of the traintrack T , we see that the subsurfaces of C and C' (approximately) corresponding to $S \setminus T$ via κ and κ' are isomorphic (Proposition 6.11 (iii)). Thus the difference of the projective structures C' and C is in their subsurfaces corresponding to T . The traintrack T is a union of branches (quadrangles), and each branch B of T corresponds to a quadrangular subsurface of C and of C' . Then, by making T sufficiently *straight* and *slim* (Definition 6.8), the product structure on the branch B yields a product-like structures on the corresponding quadrangles on C and C' that are *supported* on a round cylinder in $\hat{\mathbb{C}}$ (Definition 6.4).

Then, as in Lemma 6.5, those corresponding quadrangular subsurfaces of C and C' differ by a grafting along a multiarc M_j . Then the multiloop M in Theorem A is realized as the union of such multiarcs M_j over all branches B_j of T . In addition the number of the multiarc M_j (times 2π) is well-approximated by the difference of the weights of L and L' on B_j (Proposition 6.10 (II- iii), §6.8). Thus M is a good approximation of the difference of L and L' .

2. CONVENTIONS AND NOTATIONS

We follow the followings, unless otherwise stated:

- By a component, we mean a *connected* component.
- For a geodesic metric space X and points $x, y \in X$, we denote the geodesic segment connecting x to y by $[x, y]$.
- Let X be a manifold, and let Y be a subset of X . Then the total lift of Y is $\phi^{-1}(Y) \subset \tilde{X}$, where $\phi: \tilde{X} \rightarrow X$ is the universal covering map.
- Let X be a subset of a hyperbolic space of dimension n with its ideal boundary, $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n (\cong \mathbb{D}^n)$. Then, by $\text{Conv}(X)$, we denote the convex hull of X in \mathbb{H}^n .
- Let X be a metric space and A be a subset of X . Then, for $\epsilon > 0$, we denote the ϵ -neighborhood of A in X by $N_\epsilon(A)$ or $N_\epsilon(A, X)$.

- For $\epsilon > 0$, we say that A and B intersect ϵ -orthogonally, if the intersection angle between A and B is ϵ -close to $\pi/2$.
- By a loop, we mean a simple closed curve.

3. PRELIMINARIES

3.1. Grafting. (see also [9, 15].) Let $C = (f, \rho)$ be a projective structure on S . A loop l on C is called admissible if

- (i) $\rho(l) \in \mathrm{PSL}(2, \mathbb{C})$ is loxodromic, and
- (ii) f embeds \tilde{l} into $\hat{\mathbb{C}}$, where \tilde{l} is a lift of l to the universal cover of S .

Then, if l is admissible, then the loxodromic element $\rho(l)$ generates a finite cyclic group G in $\mathrm{PSL}(2, \mathbb{C})$. Then the limit set $\Lambda(G)$ is the fixed point set of $\rho(l)$, and G acts on the cylinder $\hat{\mathbb{C}} \setminus \Lambda(G)$ freely and property discontinuously. Thus the quotient $(\hat{\mathbb{C}} \setminus \Lambda(G))/G$ is a projective structure T_l on a torus (*Hopf torus*). Then, by (ii), l is naturally embedded in T_l . Since l is also embedded in C , there is a natural way to combine the projective structures C and T_l by cutting and pasting along l as follows. We see that $T_l \setminus l$ is a cylinder and $C \setminus l$ is a surface with two boundary components. Thus we can obtain a new projective structure on S by pairing up the boundary components of $T_l \setminus l$ and $C \setminus l$ and canonically identify them using the identification of $l \subset T$ and $l \subset C$. This surgery operation is called (2π) -grafting (of C along l), and we denote the resulting projective structure by $Gr_l(C)$. It turns out that C and $Gr_l(C)$ have the same holonomy representation ρ .

3.2. Thurston's coordinates. (see [12, 16] and also [6, 17, 3].)

We here explain more about the parametrization

$$(1) \quad \mathcal{P}(S) \cong \mathrm{Teich}(S) \times \mathcal{ML}(S).$$

3.2.1. Bending maps. Let $(\tau, L) \in \mathrm{Teich}(S) \times \mathcal{ML}(S)$. Then we regard L as a geodesic lamination on τ . Set $L = (\lambda, \mu)$, where $\lambda \in \mathcal{GL}(S)$ and μ is a transversal measure supported on λ . Let $\tilde{L} = (\tilde{\lambda}, \tilde{\mu}) \in \mathcal{ML}(\mathbb{H}^2)$ be the total lift of L to \mathbb{H}^2 . Then we can bend a copy of \mathbb{H}^2 inside \mathbb{H}^3 along $\tilde{\lambda}$ by the angle given by $\tilde{\mu}$. This map is called the bending map $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ induced by (τ, L) . Then β is well-defined up to a postcomposition with an element of $\mathrm{PSL}(2, \mathbb{C})$ (as the developing map f is). If $C = (f, \rho) \in \mathcal{P}(S)$ corresponds to $(\tau, L) \in \mathrm{Teich}(S) \times \mathcal{ML}(S)$ by (1), then we say that β is the bending map associated with C . Since the $\pi_1(S)$ -action preserves \tilde{L} , there is a representation of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{C})$ under which β is equivariant. Then this representation

is unique up to a conjugation by an element of $\mathrm{PSL}(2, \mathbb{C})$, and, indeed, this representation coincides with the holonomy representation ρ of C .

3.2.2. Maximal balls and collapsing maps. Let $C \in \mathcal{P}(S)$. Let \tilde{C} be the universal cover of $C = (f, \rho)$. An open topological ball B in \tilde{C} is called a maximal ball if the developing map $f: \tilde{C} \rightarrow \hat{\mathbb{C}}$ embeds B onto a round open ball in $\hat{\mathbb{C}}$ and there is no such a ball in \tilde{C} properly containing B . Let B be a maximal ball and let H be the hyperplane in \mathbb{H}^3 bounded by the round circle $\partial f(B)$. Recalling $\hat{\mathbb{C}} = \partial\mathbb{H}^3$, let $\Phi: f(B) \rightarrow H$ be the canonical conformal map, obtained by continuously extending of the nearest point projection of \mathbb{H}^3 onto H . Let $\partial_\infty B$ be $cl(f(B))$ minus $f(cl(B))$, where “ cl ” denotes closure; then $\partial_\infty B$ is a subset of the round circle $\partial f(B)$.

Suppose that $(\tau, L) \in \mathrm{Teich}(S) \times \mathcal{ML}(S)$ corresponds to $C = (f, \rho)$ in (1). Let $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by (τ, L) . Then there is a leaf or the closure of a component of $\mathbb{H}^2 \setminus \tilde{L}$, denoted by X , such that the bending map $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ isometrically embeds X onto $\mathrm{Conv}(\partial_\infty B) \subset H$.

Clearly $\Phi \circ f$ embeds B onto H . Then the core of the maximal ball B is the subset of B that $\Phi \circ f$ embeds onto $\mathrm{Conv}(\partial B)$, which is denoted by $\mathrm{Core}(B)$. Thus we have a unique embedding κ_B of $\mathrm{Core}(B)$ onto $X \subset \mathbb{H}^2$ such that $\beta \circ \kappa_B = \Phi \circ f$. Then, for each $x \in \mathrm{Core}(B)$, the hyperplane H is called a hyperbolic tangent plane of β at x .

It turns out that $\mathrm{Core}(B)$ are disjoint for different maximal balls B . Moreover \tilde{C} is the union of the cores $\mathrm{Core}(B)$ of all maximal balls B in \tilde{C} . Thus we can define $\tilde{\kappa}: \tilde{C} \rightarrow \mathbb{H}^2$ by $\tilde{\kappa} = \kappa_B$ on $\mathrm{Core}(B)$. Then $\tilde{\kappa}$ commutes with the action of $\pi_1(S)$ and descends to the collapsing map $\kappa: C \rightarrow \tau$. In addition, $\partial(\mathrm{Core}(B))$ is the union of disjoint copies of \mathbb{R} properly embedded in \tilde{C} . Then, by taking the union of $\partial(\mathrm{Core}(B))$ over all maximal balls B in \tilde{C} , we obtain a lamination $\tilde{\nu}$ on \tilde{C} , such that $\tilde{\kappa}$ embeds each leaf of $\tilde{\nu}$ onto a leaf of $\tilde{\lambda}$. Then the lamination $\tilde{\nu}$ descends to a lamination ν on C , and κ embeds each leaf of μ onto a leaf of λ . Then ν is called the canonical lamination corresponding to λ (via κ).

In particular the collapsing map κ has the following properties. If l is a periodic leaf of L , then $\kappa^{-1}(l)$ is a closed cylinder embedded in C . Besides this cylinder is foliated by closed leaves m of ν such that κ embeds m onto l . Then, letting \tilde{m} and \tilde{l} be corresponding lifts of m and l to \tilde{C} and \mathbb{H}^2 , respectively, f embeds \tilde{m} onto a circular arc on $\hat{\mathbb{C}}$ whose endpoints are the endpoints of the geodesic $\beta(l)$. Let h be the weight of l with respect to μ . Then $\kappa^{-1}(l)$ has a natural product structure

$\mathbb{S}^1 \times [0, h]$ so that $\mathbb{S}^1 \times \{t\}$ is a closed leaf of ν for each $t \in [0, h]$ and κ collapses $s \times [0, h]$ onto a single point on l for each $s \in \mathbb{S}^1$. Let M be the union of all periodic leaves l_1, l_2, \dots, l_n of L . Then $\kappa^{-1}(M)$ is the union of the disjoint closed cylinders $\kappa^{-1}(l_1), \dots, \kappa^{-1}(l_n)$. Then, the restriction of $\kappa: C \rightarrow \tau$ to $C \setminus \kappa^{-1}(M)$ is a diffeomorphism onto $\tau \setminus M$.

3.2.3. Thurston's metric on projective structures. We assign a canonical Euclidean metric to the cylinder $\kappa^{-1}(l_i) \cong \mathbb{S}^1 \times [0, h_i]$ for each $i = 1, 2, \dots, n$, such that its height h_i is the weight of l_i with respect to μ and that κ isometrically embeds $\mathbb{S}^1 \times \{t\}$ onto $l \subset \tau$ for each $t \in [0, h_i]$. (To be precise, for closed leaves $m = \mathbb{S}^1 \times t$ and $m' = \mathbb{S}^1 \times t'$ with $t, t' \in [0, h]$, let \tilde{m} and \tilde{m}' be corresponding lifts of m and m' , respectively, to \tilde{C} that are invariant under the same element of $\pi_1(S)$; then the circular arcs $f(\tilde{m})$ and $f(\tilde{m}')$ share their corresponding endpoints and, thus, bound a crescent region in $\hat{\mathbb{C}}$, and the angle at the vertices of the crescent region is $|t - t'| \bmod 2\pi$.) On the other hand, we assign $C \setminus \kappa^{-1}(M)$ the pullback hyperbolic metric of τ via κ . Thus we have defined a piecewise Euclidean/hyperbolic metric on C , which is called Thurston's metric.

We last see that the transversal measure μ supported on λ naturally induces a transversal measure on ν . On each Euclidean cylinder $\kappa^{-1}(l_i) = \mathbb{S}^1 \times [0, h_i]$ in C , we assign a unique transversal measure to ν by the difference in the second coordinate $[0, h_i]$, so that the total measure of the cylinder is the weight of l_i with respect to μ . Since κ isometrically embeds the hyperbolic subsurface of C onto $\tau \setminus M$, we can isomorphically pull back $L \setminus M$ to a measured lamination on $C \setminus \kappa^{-1}(M)$ via κ . The union of those measured laminations on the Euclidean and hyperbolic subsurfaces of C is called the canonical measured lamination on C corresponding to L .

3.3. Intersection for measured laminations. (see [3])

Let L be a measured geodesic lamination on \mathbb{H}^2 . Let X be a subset of \mathbb{H}^2 .

Definition 3.1. *The intersection of L and X is the minimal sublamination of L that contains all leaves of L intersecting X . We denote this intersection by $I(L, X)$.*

3.4. Isomorphism via developing maps.

Definition 3.2. *Let $C = (f, \rho)$ and $C' = (f', \rho)$ be projective structures on a surface F with the same holonomy ρ , where $f: \tilde{C} \rightarrow \hat{\mathbb{C}}$ and $f': \tilde{C}' \rightarrow \hat{\mathbb{C}}$ are their developing maps. Then C and C' are*

isomorphic via f and f' , if there is a marking preserving homeomorphism $\phi: C \rightarrow C'$, such that, letting $\tilde{\phi}: \tilde{C} \rightarrow \tilde{C}'$ be the lift of ϕ , we have $f = f' \circ \tilde{\phi}: \tilde{C} \rightarrow \hat{\mathbb{C}}$. Then, we also say that the isomorphism ϕ is compatible with f and f' .

3.5. Equivariant homotopies between Bending maps.

Lemma 3.3. *Let $C_1 = (\tau_1, \lambda_1)$ and $C_2 = (\tau_2, \lambda_2)$ be projective structures on S with the same holonomy $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Let $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ and $\beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending maps associated with C and C' , respectively. Then β and β' are ρ -equivariantly homotopic (note that the domains of β and β' are identified via the markings $\psi_1: S \rightarrow \tau_1$ and $\psi_2: S \rightarrow \tau_2$ since $\tau_1, \tau_2 \in \mathrm{Teich}(S)$).*

Proof. Via the markings ψ_1 and ψ_2 , we regard a subset of S also as a subset of τ_1 and τ_2 .

Step 1. We first construct an equivariant homotopy for corresponding loops on τ_1 and τ_2 . Let l be a loop on S . Let $\tilde{l} (\cong \mathbb{R})$ be a lift of l to the universal cover \tilde{S} of S . Then $\beta_1|_{\tilde{l}}$ and $\beta_2|_{\tilde{l}}$ are equivariant under the restriction of ρ to $\langle l \rangle \subset \pi_1(S)$, the infinite cyclic group generated by $l \in \pi_1(S)$. Then $\rho(l)$ may be of any type of hyperbolic isometry, i.e. parabolic, elliptic, or loxodromic. In either case, we can easily construct a homotopy between $\beta_1|_{\tilde{l}}$ and $\beta_2|_{\tilde{l}}$ that is equivariant under $\rho|_{\langle l \rangle}$.

Step 2. Next let P be a pair of pants embedded in S . Let l_1, l_2, l_3 be the boundary loops of P . Pick disjoint arcs a_1, a_2, a_3 properly embedded in P that decompose P into two hexagons. Let \tilde{P} be a lift of P to \tilde{S} . Then we show that there is a homotopy between $\beta_1|_{\tilde{P}}$ and $\beta_2|_{\tilde{P}}$ equivariant under $\rho|_{\pi_1(P)}$. For each $j = 1, 2, 3$, pick a lift \tilde{l}_j of l_j to \tilde{P} . Then, by Step 1, we have a homotopy connecting $\beta_1|_{\tilde{l}_j}$ and $\beta_2|_{\tilde{l}_j}$ that is equivariant under $\rho|_{\pi_1(l_j)}$. By equivariantly extending those homotopies, we have a homotopy $\Phi_{\partial\tilde{P}}: \partial\tilde{P} \times [0, 1] \rightarrow \mathbb{H}^3$ between $\beta_1|_{\partial\tilde{P}}$ and $\beta_2|_{\partial\tilde{P}}$ that is $\rho|_{\pi_1(P)}$ -equivariant. Then we can easily extend this homotopy $\Phi_{\partial\tilde{P}}$ to the homotopy between the lifts of arcs a_i ($i = 1, 2, 3$) to \tilde{P} and then to the entire lift \tilde{P} , so that the extension is equivariant under $\rho|_{\pi_1(P)}$.

Step 3. Pick a maximal multiloop M on S , which decompose S into $2(g-1)$ pairs of pants P_k ($k = 1, 2, \dots, 2(g-1)$). Let \tilde{M} denote the total lift of M to \tilde{S} . Then we can obtain a ρ -equivariant homotopy $\Phi_{\tilde{M}}$ between $\beta_1|_{\tilde{M}}$ and $\beta_2|_{\tilde{M}}$ similarly as we obtained the homotopy $\Phi_{\partial\tilde{P}}$ in Step 2. For arbitrary $k \in \{1, 2, \dots, 2(g-1)\}$, let \tilde{P}_k be a lift of P_k to \tilde{S} . Then $\Phi_{\tilde{M}}$ induces a homotopy $\Phi_{\partial\tilde{P}_k}$ between $\beta_1|_{\partial\tilde{P}_k}$ and

$\beta_2|_{\partial\tilde{P}_k}$. Similarly to Step 2, we can extend this induced homotopy to a homotopy $\Phi_{\tilde{P}_k}$ between $\beta_1|_{\tilde{P}_k}$ and $\beta_2|_{\tilde{P}_k}$ that is equivariant under $\rho|_{\pi_1(P_k)}$. Since $\Phi_{\tilde{P}_k}$ and $\Phi_{\tilde{M}}$ coincide on $\partial\tilde{P}_k$ and $\Phi_{\tilde{M}}$ is ρ -equivariant, We can ρ -equivalently extend all homotopies $\Phi_{\tilde{P}_k}$ ($k = 1, 2, \dots, 2(g-1)$) between $\beta_1|_{\tilde{P}_k}$ and $\beta_2|_{\tilde{P}_k}$ and obtain a desired homotopy between β_1 and β_2 . \square

3.6. Quasi-isometries and bilipschitz maps.

Definition 3.4. *Let X and Y are metric space. Let $f, g: X \rightarrow Y$ be continuous maps. We say f is an ϵ -perturbation of g for $\epsilon > 0$, if $\text{dist}_Y(f(x), g(x)) < \epsilon$ for all $x \in X$. If, in addition, f and g are homotopic, then we say that f and g are ϵ -homotopic.*

Lemma 3.5. *Let τ be a closed hyperbolic surface. For every $\epsilon > 0$, there exists $D_\tau > 0$, such that, if $\eta: \tau \rightarrow \tau'$ is a $(1 + \delta, \delta)$ -quasiisometry onto a hyperbolic surface τ' with $0 < \delta < D_\tau$, then we can ϵ -homotope η to a smooth $(1 + \epsilon)$ -bilipschitz map.*

Proof. (c.f. [15, the proof of Theorem 7.2].) *Step1.* Pick $\tau \in \text{Teich}(S)$. Fix a closed hyperbolic surface τ homeomorphic to S . First consider a simple geodesic segment $[A, B]$ on τ that connects distinct points A and B on τ . For every $\epsilon > 0$, assuming that $\delta > 0$ is sufficiently small, if $\eta: \tau \rightarrow \tau'$ is a $(1 + \delta, \delta)$ -quasiisometry with $\tau' \in \text{Teich}(S)$, then we can ϵ -homotope η to $\eta': \tau \rightarrow \tau'$ so that $\eta(A) = \eta'(A)$, $\eta(B) = \eta'(B)$, and $\eta'|_{[A, B]}$ is a $(1 + \epsilon)$ -bilipschitz embedding onto a geodesic segment connecting $\eta(A)$ to $\eta(B)$.

Step 2. Next consider a finite triangulation $\Delta = \{\Delta_i\}$ ($i = 1, 2, \dots, n$) of τ , where Δ_i are hyperbolic triangles on τ with disjoint interiors such that $\cup_i \Delta_i = \tau$. Since there are only finitely many edges of Δ , for every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then given a $(1 + \delta, \delta)$ -quasiisometry $\eta: \tau \rightarrow \tau'$ onto a hyperbolic surface τ' , we can ϵ -homotope η to $\eta': \tau \rightarrow \tau'$ satisfying the property of η' in Step 1 for all edges of the triangulation Δ . Then, by taking sufficiently small $\delta > 0$, we can in addition assume that η' is a $(1 + \epsilon)$ -bilipschitz embedding when restricted to each Δ_i . Therefore, since the triangulation is locally finite, if $\delta > 0$ is sufficiently small, then η is ϵ -homotopic to a smooth $(1 + \epsilon)$ -bilipschitz map. \square

4. BILIPSCHITZ CURVES AND BENDING MAPS

Definition 4.1. *Let λ and λ' be (possibly measured) geodesic laminations on a hyperbolic surface τ . Then the angle between λ and λ'*

is

$$\angle(\lambda, \lambda') := \sup \angle_p(l, l') \in [0, \pi/2],$$

over all $p \in \lambda \cap \lambda'$, where l and l' are the (non-oriented) leaves of λ and λ' , respectively, such that $p \in l \cap l'$.

The following proposition is the main statement of this section:

Proposition 4.2. *For every $\delta > 0$, there exists $\epsilon > 0$ such that, if $L = (\lambda, \mu)$ is a measured geodesic lamination on \mathbb{H}^2 with $\text{Area}_{\mathbb{H}^2}(\lambda) = 0$ and l is a geodesic on \mathbb{H}^2 with $\angle(l, L) < \epsilon$, then, letting $\beta = \beta_L: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map associated with L ,*

(i) $\beta_L|l: l \rightarrow \mathbb{H}^3$ is a $(1 + \delta)$ -bilipschitz embedding,

and, letting m be the geodesic in \mathbb{H}^3 connecting the endpoints of $\beta_L(L)$,

(ii) for each point $x \in l$, $\beta_L(x)$ is δ -close to m , and, if β_L is differentiable at x , then the tangent vector of $\beta_L|l$ at x is δ -parallel to m ,

that is, the tangent vector of $\beta_L|l$ in \mathbb{H}^3 at x is δ -orthogonal to the totally geodesic hyperplane that is orthogonal to m and contains $\beta(x)$.

Remark 4.3. *The condition on $\angle(l, L)$ is new. Similar statements are in [5, 7, 3].*

Letting $\Phi_m: \mathbb{H}^3 \rightarrow m$ be the nearest point projection, we in addition have

Corollary 4.4. *(iii) $\Phi_m \circ \beta_L|l: l \rightarrow m$ is a $(1 + \delta)$ -bilipschitz map.*

Proof. For each point $y \in m$, $\Phi_m^{-1}(y)$, is a totally geodesic hyperplane in \mathbb{H}^3 orthogonal to m . Then \mathbb{H}^3 is foliated by such hyperplanes. Since $\text{Area}_{\mathbb{H}^2}(\lambda) = 0$ is $\beta|l$ is differentiable almost everywhere. By (ii), the curve $\beta_L|l$ stays in a small neighborhood of m and δ -orthogonally intersects this foliation of \mathbb{H}^3 at almost every point at x . By taking sufficiently small $\epsilon > 0$, we have the corollary. \square

We first prove a similar statements for geodesic segments of bounded lengths:

Proposition 4.5. *For every (large) $K > 0$ and (small) $\delta > 0$, there exists $\epsilon > 0$ with the following properties:*

(i) *If L is a measured geodesic lamination and $l (\cong \mathbb{R})$ is a geodesic on \mathbb{H}^2 with $\angle(l, L) < \epsilon$, then, if points x, y on l satisfies $x < y$ and $\text{dist}_{\mathbb{H}^2}(x, y) < K$, then we have $(1 - \delta) \cdot \text{dist}_{\mathbb{H}^2}(x, y) < \text{dist}_{\mathbb{H}^3}(\beta_L(x), \beta_L(y))$.*

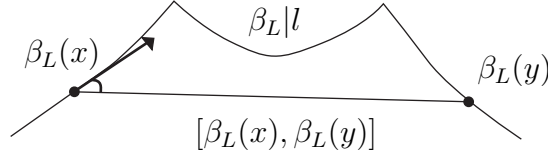


FIGURE 1.

- (ii) If $\beta|l$ is differentiable at x , for $y \in l$ with $y \neq x$, let $\theta_y(x) \in [0, \pi]$ denote the angle between the oriented geodesic segment $[\beta_L(x), \beta_L(y)]$ and the tangent vector $(\beta_L|l)'(x) \in T_{\beta(x)}(\mathbb{H}^3)$. Then $\theta_y(x) < \delta$ for all $x, y \in l$ with $\text{dist}(x, y) < K$ such that $\theta_y(x)$ is well-defined (see Figure 1).

Proof. First consider a right hyperbolic triangle $\triangle ABC$ in \mathbb{H}^2 (with geodesic edges) with $\angle C = \pi/2$, where A, B, C are the vertices of the triangle. Then by elementary trigonometry, we can easily prove:

Lemma 4.6. *For every (small) $\delta' \in (0, 1]$, there exists $\epsilon > 0$ depending only on K , such that, if $\angle B < \epsilon$ and $\text{length}(AB) < K$, then*

$$(1 - \delta') \cdot \text{length}(AB) < \text{length}(CA) - \text{length}(BC),$$

and, for every point A' in \mathbb{H}^2 with $\text{dist}(C, A') < \text{dist}(C, A)$, we have

$$\angle A'BC < \delta'.$$

Pick $K > 0$ and $\delta \in (0, 1]$. Then pick $\delta' > 0$ with $\delta' < \delta$. Let $\epsilon > 0$ be the number obtained by applying Lemma 4.6 for K and δ' . Then we can assume that $\delta' + \epsilon < \delta$, if necessarily by taking smaller $\epsilon > 0$.

Let L be a measured (geodesics) lamination and $l (\cong \mathbb{R})$ be a geodesic on \mathbb{H}^2 with $\angle(l, L) < \epsilon$. Let x and y be distinct points on l within distance less than K such that $x < y$. Let $I \in \mathcal{ML}(\mathbb{H}^2)$ be the intersection $I([x, y], L)$ of L and the geodesic segment $[x, y]$ connecting x to y (see §3.3). We can assume that $[x, y]$ intersects at least one leaf of L transversally, since otherwise Proposition 4.5 clearly holds for such x, y . Let m denote the leaf of I closest to x . Then, there is a unique point $z \in \mathbb{H}^2$ such that $\triangle xyz$ is a hyperbolic right triangle with $\angle z = \pi/2$ and, letting $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the orientation preserving isometry preserving l and taking $l \cap m$ to x , we have $[x, z] \subset \eta(m)$. Then $[x, z]$ is disjoint from I , unless m contains x . Thus, since $\angle(l, L) < \epsilon$, we have $\angle(yxz) < \epsilon$. Let $\beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map induced by I . Then β_I isometrically embeds $[x, z]$ into \mathbb{H}^3 . Therefore $\text{dist}_{\mathbb{H}^3}(\beta_I(x), \beta_I(z)) = \text{dist}_{\mathbb{H}^2}(x, z)$. Since bending maps are 1-lipschitz,

$\text{dist}(\beta_I(z), \beta_I(y)) \leq \text{dist}(z, y)$. By the triangle inequality, we have

$$\begin{aligned} \text{dist}(\beta_I(x), \beta_I(y)) &\geq \text{dist}(\beta_I(x), \beta_I(z)) - \text{dist}(\beta_I(z), \beta_I(y)) \\ &\geq \text{dist}(x, z) - \text{dist}(z, y). \end{aligned}$$

Then, by the first conclusion of Lemma 4.6, we have

$$\text{dist}(\beta_I(x), \beta_I(y)) > (1 - \delta') \cdot \text{dist}(x, y).$$

Since $\beta_I = \beta_L$ on $[x, y]$, $\text{dist}(\beta_L(x), \beta_L(y)) > (1 - \delta') \cdot \text{dist}(x, y)$; thus we have shown i(i).

By applying the second conclusion of Lemma 4.6 to $\triangle A'BC = \triangle \beta_I(y)\beta_I(x)\beta_I(z)$, we have $\angle \beta_I(y)\beta_I(x)\beta_I(z) < \delta'$. By the triangle inequality, $\theta_y(x) \leq \angle yxz + \angle \beta_I(y)\beta_I(x)\beta_I(z) < \epsilon + \delta' < \delta$. Thus we have proved (ii). 4.5

Proof of Proposition 4.2. Set $\beta = \beta_I$ (i) Let $L = (\lambda, \mu)$ be a geodesic lamination on \mathbb{H}^2 and let $l \cong \mathbb{R}$ be a geodesic on \mathbb{H}^2 . Let x, y be different points on l with $x < y$ such that $\beta|l$ is differentiable at y . Then, as before, let $\theta_x(y) \in [0, \pi]$ denote the angle between the tangent vector of $\beta|l$ at y and the oriented geodesic $[\beta(x), \beta(y)]$ in \mathbb{H}^3 .

Fixing sufficiently small $\delta' > 0$, we show that there exists $\epsilon > 0$, depending only on δ' , such that if $\angle(l, L) < \epsilon$, then $\theta_x(y) < \delta'$. Pick $K > 0$ and $\delta'' > 0$ with $\delta'' < \delta'/2$. Let $\epsilon' = \epsilon'(K, \delta'') > 0$ be the number obtained by applying Lemma 4.5 to K and δ'' . Then divide the closed geodesics $[x, y]$ into subintervals $[p_0, p_1], [p_1, p_2], \dots, [p_{n-1}, p_n]$ and $x = p_0 < p_1 < \dots < p_n = y$ so that

- (1) p_i are not on the leaves of L for $i = 1, 2, \dots, n-1$, and
- (2) $K/2 < \text{dist}_{\mathbb{H}^2}(p_i, p_{i+1}) < K$ for $i = 0, 1, \dots, n-2$

(recall that $\text{Area}_{\mathbb{H}^2}(\lambda) = 0$). Consider the piecewise geodesic curve $\cup_{i=0}^{n-1} [\beta(p_i), \beta(p_{i+1})]$ in \mathbb{H}^3 , which connects $\beta(x)$ to $\beta(y)$. By Lemma 4.5 (ii), we have $\theta_{p_{n-1}}(y) < \delta''$ and also $\pi - \angle(\beta(p_{i-1}), \beta(p_i), \beta(p_{i+1})) < 2\delta''$ for all $i = 1, 2, \dots, n-1$. By Lemma 4.5 (i), $\text{length}([\beta(p_i), \beta(p_{i+1})]) > (1 - \delta'') \cdot (K/2)$ for $i = 0, 1, \dots, n-2$. Thus, retaking sufficiently small $\delta' > 0$ if necessarily, the piecewise geodesic curve $\cup_{i=0}^{n-1} [\beta(p_i), \beta(p_{i+1})]$ has sufficiently long segments (except $[\beta(p_{n-1}), \beta(p_n)]$) and the bending angle at its singularity points are sufficiently small. Therefore, assuming that $\delta'' > 0$ is sufficiently small, we have $\angle \beta(x)\beta(y)\beta(p_{n-1}) < \delta'/2$ (see [5, §I.4.2]; also [7, 3]). Then, by the triangle inequality,

$$0 < \theta_x(y) < \angle \beta(p_{n-1})\beta(y)\beta(x) + \theta_{p_{n-1}}(y) < \delta'/2 + \delta''.$$

Thus we have that $0 < \theta_x(y) < \delta'$. We have

$$\frac{d \text{length}([\beta(x), \beta(y)])}{dy} = \cos(\theta_x(y))$$

(see [5, §I.4.2]; also [7, 3]). Then, by taking sufficiently small $\delta' > 0$ if necessary, we have $\frac{1}{1+\delta} < \cos(\theta_x(y)) \leq 1$ for all different x, y on l such that $\beta|l$ is differentiable at y . Since $\beta|l$ is differentiable at almost all points of l , $\beta|l$ is a $(1+\delta)$ -bilipschitz embedding.

(ii) Pick $\delta' > 0$ with $2\delta' < \delta$. Then, in the proof of (i), we have shown that there exists $\epsilon > 0$, such that, if $\angle(l, L) < \epsilon$, then $\theta_x(y) < \delta'$ for all different $x, y \in l$ such that $\beta|l$ is differentiable at y . Taking the limits as x goes to the end points $\pm\infty$ of $l \cong \mathbb{R}$, we have $\theta_{-\infty}(y), \theta_{\infty}(y) \leq \delta'$. Thus we have $\beta(-\infty)\beta(y)\beta(\infty) > \pi - 2\delta'$, where $\beta(-\infty), \beta(\infty) \in \hat{\mathbb{C}}$ are the ideal endpoints of the geodesic m bounded distance away from the bilipschitz curve $\beta|l$. It is well-known that the area of a triangle in \mathbb{H}^2 is equal to π minus the sum of the angles of the vertices. Thus $\text{Area}_{\mathbb{H}^2}(\triangle\beta(-\infty)\beta(y)\beta(\infty)) < 2\delta'$. Thus, if necessary by taking smaller $\delta' > 0$, we can assume that $\text{dist}_{\mathbb{H}^3}(\beta(y), m) < \delta$. Recalling that $\Phi_m: \mathbb{H}^3 \rightarrow m$ is the nearest point projection, $\triangle\beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y)$ has area less than δ' . Using the same area formula, we have $\angle\beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y) < \pi/2 - \delta'$. Since $\theta_{-\infty}(y) < \delta'$ and $2\delta' < \delta$, by the triangle inequality, we see that the tangent vector of $\beta|l$ at y is δ -parallel to m . (Alternatively, the δ -closeness is immediate from (i) and Morse Lemma.) 4.2

5. LOCAL STABILITY OF BENDING MAPS IN \mathcal{PML}

5.1. Bending maps with a fixed bending lamination.

Proposition 5.1. *Let $C = (\tau, L)$ and $C' = (\tau', L')$ be projective structures on S with fixed holonomy ρ . Let β and $\beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending maps associated with C and C' , respectively. Then, if $[L] = [L'] \in \mathcal{PML}(S)$, we have $\beta = \beta'$.*

Proof. We first show that there exists a continuous shearing map $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that $\beta \circ \eta = \beta'$. Let \tilde{L} and \tilde{L}' be the total lifts of L and L' to \mathbb{H}^2 , respectively. Since $[L] = [L']$ in $\mathcal{PML}(S)$, we have $[\tilde{L}] = [\tilde{L}'] \in \mathcal{PML}(\tilde{S})$. Then

Lemma 5.2. *Let l and l' be corresponding leaves of \tilde{L} and \tilde{L}' , respectively. In addition, assign the same orientation to l and l' . Then $\beta(l) = \beta'(l')$ as oriented geodesics in \mathbb{H}^3 ; thus there exists a unique orientation-preserving isometry $\eta_l: l \rightarrow l'$ such that $\beta = \beta' \circ \eta_l$ on l .*

Proof. Let l and l' be as in the assumption. Then β and β' isometrically embed l and l' , respectively, onto a geodesic in \mathbb{H}^3 . By Lemma 3.3, $\beta, \beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3$ are ρ -equivariantly homotopic. Thus the geodesic $\beta(l)$ and $\beta(l')$ has bounded Hausdorff distance, and therefore $\beta(l) = \beta(l')$. Hence, there is an isometry $\eta_l: l \rightarrow l'$ such that $\beta = \beta' \circ \eta_l$. Since β and β' are ρ -equivariantly homotopic, that corresponding endpoints of l and l' (in $\partial\tilde{S}$) map to the same point on $\hat{\mathbb{C}}$. Thus η_l preserves the orientation. \square

Corollary 5.3. *Let P and P' are the closures of corresponding components of $\mathbb{H}^2 \setminus \tilde{\lambda}$ and $\mathbb{H}^2 \setminus \tilde{\lambda}'$, respectively. Then there exists a unique orientation preserving isometry $\eta_P: P \rightarrow P'$ such that $\beta' \circ \eta_P = \beta$ on P .*

Proof. The equality $[L] = [L']$ induces an orientation-preserving homeomorphism between P and P' and, in particular, between ∂P and $\partial P'$. By Lemma 5.2, for corresponding boundary geodesics l and l' of P and P' , respectively, we have an orientation preserving isometry $\eta_l: l \rightarrow l'$ such that $\beta' \circ \eta_l = \beta$ on l . Then define $\eta_\partial: \partial P \rightarrow \partial P'$ by $\eta_\partial = \eta_l$ on each boundary geodesic l of P . The bending maps β and β' isometrically embed P and P' , respectively, into the same totally geodesic hyperplane in \mathbb{H}^3 . Thus $\eta_\partial: \partial P \rightarrow \partial P'$ uniquely extends to an isometry $\eta: P \rightarrow P'$ such that $\beta' \circ \eta = \beta$ on P . \square

The hyperbolic plane \mathbb{H}^2 decomposes into components of $\mathbb{H}^2 \setminus \tilde{L}$ and leaves of \tilde{L} . Thus we can define $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ by $\eta(x) = \eta_l(x)$ if x is a leaf l of L as in Lemma 5.2 and by $\eta(x) = \eta_P(x)$ if x is in the closure of a component P of $\mathbb{H}^2 \setminus L$ as in Corollary 5.3. Then, on isolated leaves of \tilde{L} , η is well-defined and continuous by the uniqueness of η_P . Thus $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is well-defined, and $\beta = \beta' \circ \eta$. The bending maps β and β' are continuous, and moreover they are locally injective in the complement of isolated leaves of weight π modulo 2π . Therefore, η must be continuous since $\beta = \beta' \circ \eta$. Thus $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is a shearing map along \tilde{L} minus isolated leaves of L .

Last we show that the shearing map $\eta: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is an isometry. Then it suffice to show that for each $x \in \mathbb{H}^2$ contained in an irrational leaf of \tilde{L} , there exists a small neighborhood U of x such that $\eta|_U: U \rightarrow \mathbb{H}^2$ is an isometric embedding (i.e. there is no nontrivial shearing along leaves of \tilde{L} that intersects U). Let U be a neighborhood of x such that $\beta|_U: U \rightarrow \mathbb{H}^3$ is injective. Then $\beta|_U$ is an isometric embedding onto its image $\beta(U)$ with the intrinsic metric induced from \mathbb{H}^3 . Then $\beta|_U$ is an isometric embedding onto its image $\beta(U)$ with the intrinsic metric induced from \mathbb{H}^3 . Then, since $\beta = \beta' \circ \eta$, β' is injective on $\eta(U)$ and

it isometrically embeds $\eta(U)$ onto $\beta(U)$. Therefore η is isometry on U .

5.1

5.2. Stability of bending maps with converging bending laminations. We fix a representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Let $(C_i)_{i \in \mathbb{N}}$ be a sequence in \mathcal{P}_ρ . For each $i \in \mathbb{N}$, set $C_i = (\tau_i, L_i) \in \mathrm{Teich}(S) \times \mathcal{ML}(S)$ and $L_i = (\lambda_i, \mu_i)$, where λ_i is in $\mathcal{GL}(\tau_i)$ and μ_i is a transversal measure supposed on λ_i . Let $\beta_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be the bending map associates with C_i . In Proposition 5.1, we have shown that, if the projective measured laminations of two projective structures coincide, then so do the bending maps. Our main goal of §5 is generalize Proposition 5.1 to the case that a sequence of projective measured laminations converges in $\mathcal{PML}(S)$:

Theorem 5.4. *Suppose that $\lim_{i \rightarrow \infty} [L_i] = [L]$ in $\mathcal{PML}(S)$. Then β_i converges to β as $i \rightarrow \infty$. More precisely, we have*

- (i) *For every $\epsilon > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then there exists a marking preserving $(1+\epsilon)$ -bilipschitz map $\eta_i: \tau \rightarrow \tau_i$ such that $\beta_i \circ \tilde{\eta}_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is ϵ -close to β in C^0 -topology, where $\tilde{\eta}_i: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is the lift of η_i .*
- (ii) *Suppose, in addition, that λ_i converges to a lamination λ_∞ in $\mathcal{GL}(S)$ as $i \rightarrow \infty$ with the Chabauty topology. Then, we can in addition assume that $\beta_i \circ \tilde{\eta}_i$ is ϵ -close to β in C^1 -topology in the compliment of the ϵ -neighborhood of $\tilde{\lambda}_\infty$, the total lift of λ_∞ to \mathbb{H}^2 .*

We first generalize Lemma 5.2 to the setting of Theorem 5.4. From now on, we assume that λ_i converges to a lamination on λ_∞ in $\mathcal{GL}(S)$. Then, since $[L_i] \rightarrow [L]$ in $\mathcal{PML}(S)$, λ is a sublamination of λ_∞ . Since $\tau, \tau_i \in \mathrm{Teich}(S)$, therefore $\mathcal{GL}(S)$, $\mathcal{GL}(\tau)$ and $\mathcal{GL}(\tau_i)$ are canonically identified. Then, for each $i \in \mathbb{N}$, let ν_i be the geodesic lamination on τ corresponding to λ_i on τ_i , and let ν_∞ be the geodesic lamination on τ corresponding to λ_∞ on S . Then ν_i converges to ν_∞ as $i \rightarrow \infty$ in the Chabauty topology. Letting $\tilde{\nu}_i \in \mathcal{GL}(\mathbb{H}^2)$ denote the total lift $\nu_i \in \mathcal{GL}(\tau_i)$ for each $i \in \mathbb{N}$, we have

Lemma 5.5. *For every $\epsilon > 0$, there exists $I \in \mathbb{N}$ such that, if $i > I$, then $\beta|_l$ is a $(1+\epsilon)$ -bilipschitz embedding for all leaves l of $\tilde{\nu}_i$.*

Proof. For every $\delta > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then $\angle_\tau(\nu_i, \nu_\infty) < \delta$. Let $\tilde{\lambda} \in \mathcal{GL}(\mathbb{H}^2)$ be the total lift of $\lambda \in \mathcal{GL}(\tau)$. In addition, by Lemma 4.2 (i), for every $\epsilon > 0$, there exists $\delta > 0$, such that if a geodesic l on \mathbb{H}^2 , in particular a leaf l of $\tilde{\nu}_i$, satisfies $\angle_{\mathbb{H}^2}(l, \tilde{\lambda}) < \delta$,

then $\beta|l$ is a $(1 + \epsilon)$ -bilipschitz embedding. This completes the proof. \square

We next prove Theorem 5.4 (i) except the bilipschitz property, still assuming the convergence $\lambda_i \rightarrow \lambda_\infty$:

Proposition 5.6. *Every $\epsilon > 0$, there exists $I \in \mathbb{N}$, if $i > I$, then there exists a marking preserving homeomorphism $\eta_i: \tau \rightarrow \tau_i$ such that β and $\beta_i \circ \tilde{\phi}_i$ are ϵ -close in C^0 -topology.*

Proof. Step 1. We first define an appropriate homeomorphism $\eta_i: \nu_i \rightarrow \lambda_i$ for sufficiently large $i \in \mathbb{N}$. By Lemma 5.5, for every $\epsilon > 0$, there exists $I \in \mathbb{N}$ such that, if $i > I$, then, for every leaf l of $\tilde{\nu}_i$, $\beta|l$ is a $(1 + \epsilon)$ -bilipschitz embedding. Let $\tilde{\lambda}_i \in \mathcal{GL}(\mathbb{H}^2)$ be the total lift of $\lambda_i \in \mathcal{GL}(\tau_i)$ for each $i \in \mathbb{N}$, and let $m (= m_l)$ be the leave of $\tilde{\lambda}_i$ corresponding to l . Then, by Lemma 4.2 (ii), we can in addition assume that the bilipschitz curve $\beta|l$ is contained in the ϵ -neighborhood of the geodesic $\beta_i(m)$ in \mathbb{H}^3 . Let $\Phi_{\beta_i(m)}: \mathbb{H}^3 \rightarrow \beta_i(m)$ be the nearest point projection. Then, by Corollary 4.4, for every $\epsilon > 0$, if i is sufficiently large, then, $\Phi_{\beta_i(m)} \circ \beta|l: l \rightarrow \beta_i(m)$ is a $(1 + \epsilon)$ -bilipschitz map for all leaves l of $\tilde{\nu}_i$. Thus there is a unique bilipschitz map $\tilde{\eta}_l: l \rightarrow m$ such that $\Phi_{\beta_i(m)} \circ \beta|l = \beta_i \circ \tilde{\eta}_l$. Then, for each $x \in l$, $\text{dist}_{\mathbb{H}^3}(\beta(x), \beta_i \circ \tilde{\eta}_l(x)) < \epsilon$. Thus we can define a homeomorphism $\tilde{\eta}_i: \tilde{\nu}_i \rightarrow \tilde{\lambda}_i$ by $\tilde{\eta}_i(x) = \tilde{\eta}_l(x)$, where l is a leaf l of $\tilde{\nu}_i$ containing x . Then $\beta_i \circ \tilde{\eta}_i$ and $\beta|\tilde{\nu}_i$ are ϵ -close pointwise. Since β and β_i are ρ -equivariant, the action of $\pi_1(S)$ commutes with $\tilde{\eta}_i$. Thus $\tilde{\eta}: \tilde{\nu}_i \rightarrow \tilde{\lambda}_i$ descends to a homeomorphism $\eta_i: \nu_i \rightarrow \lambda_i$.

Step 2. We next extend $\eta_i: \nu_i \rightarrow \lambda_i$ to a desired homeomorphism $\eta_i: \tau \rightarrow \tau_i$. Let P be a component of $\mathbb{H}^2 \setminus \tilde{\nu}_i$.

Lemma 5.7. *For every $\epsilon > 0$, there exists $I \in \mathbb{N}$, such that, for all components P of $\mathbb{H}^2 \setminus \tilde{\nu}_i$, $\beta|P$ is a $(1 + \epsilon, \epsilon)$ -quasiisometric embedding.*

Proof. For a geodesic segment s on \mathbb{H}^2 , let s_ϵ denote s minus the $(\epsilon/2)$ -neighborhood of the endpoints of s , so that s_ϵ is a subsegment of s whose length is $\text{length}(s) - \epsilon$. Recall that ν_i converges to ν_∞ as $i \rightarrow \infty$ in $\mathcal{GL}(\tau)$ and λ is a sublamination of ν_∞ . Thus, for every $\epsilon > 0$, for sufficiently large $i \in \mathbb{N}$, λ is contained in the ϵ -neighborhood ν_i . Therefore, for every $\epsilon > 0$, if i is sufficiently large, then, for every geodesic segment s in $\mathbb{H}^2 \setminus \tilde{\nu}_i$, we have $\angle(s_\epsilon, \lambda) < \epsilon$. Thus, by Proposition 4.2 (i), for arbitrary $\epsilon > 0$, if i is sufficiently large, then for every geodesic s in $\mathbb{H}^2 \setminus \tilde{\nu}_i$, $\beta|s_\epsilon$ is a $(1 + \epsilon)$ -bilipschitz map. Therefore $\beta|s$ is a $(1 + \epsilon, \epsilon)$ -quasiisometric embedding. \square

Let P be a component of $\mathbb{H}^2 \setminus \tilde{\nu}_i$, and let Q be the component of $\mathbb{H}^2 \setminus \tilde{\lambda}_i$ corresponding to P . Noting that β_i isometrically embeds Q into a totally geodesic hyperplane in \mathbb{H}^3 , let $\Phi_Q: \mathbb{H}^3 \rightarrow \beta_i(Q)$ denote the nearest point projection. By Lemma 3.3 and Lemma 5.7, we see that, for every $\epsilon > 0$, if i is sufficiently large, then $\Phi_Q \circ \beta|_P: P \rightarrow \beta_i(Q) \cong Q$ is a $(1 + \epsilon, \epsilon)$ -quasiisometry for all components P of $\mathbb{H}^2 \setminus \tilde{\nu}_i$. Let $\tilde{\eta}_P: P \rightarrow Q$ denote this quasiisometry. Then, we can in addition assume that $\beta_i \circ \tilde{\eta}_P$ and $\beta|_P$ are ϵ -close pointwise for all components P . Since β and β_i are ρ -equivariant, letting $Stab(P)$ be the maximal subgroup of $\pi_1(S)$ that preserves $P \subset \mathbb{H}^2$, $Stab(P)$ commutes with $\tilde{\eta}_P$. Since β_i isometrically embeds Q , for every $\epsilon > 0$, if i is sufficiently large, then, for every component P of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ and every boundary geodesic l of P , $\tilde{\eta}_P|_l: l \rightarrow Q$ above and $\tilde{\eta}_l: l \rightarrow m$ in Step 1 are ϵ -close pointwise. Therefore, if i is sufficiently large, we can perturb $\tilde{\eta}_P$ so that $\tilde{\eta}_P: P \rightarrow Q$ is a homeomorphism and $\tilde{\eta}_P|_l = \tilde{\eta}_l$ for each boundary component l of P , but preserving the properties that $\beta_i \circ \tilde{\eta}_P$ and $\beta|_P$ are ϵ -close pointwise and that $\tilde{\eta}_P$ commutes with $Stab(P)$. Therefore we can continuously extend a homeomorphism $\tilde{\phi}_i: \tilde{\nu}_i \rightarrow \tilde{\lambda}_i \subset \mathbb{H}^2$ to a homeomorphism $\tilde{\eta}_i: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ by $\tilde{\eta}_i|_P = \tilde{\eta}_P$ for all components P of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ so that $\tilde{\eta}_i$ commutes with the action of $\pi_1(S)$. Thus $\beta_i \circ \tilde{\eta}_i$ and β are ϵ -close, and $\tilde{\eta}_i$ descends to the desired homeomorphism $\eta: \tau \rightarrow \tau'$.

5.6

5.3. Regular neighborhoods of bending maps. Bending maps are in general (highly) non-injective. However, as a substitute, we construct a “regular neighborhood” of a bending map $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$.

Case 1. Suppose that L contains no leaf of weight π modulo 2π . Then β is locally injective. Let M be the sublamination of L consisting of the periodic leaves of L , and let $\tilde{M} \in \mathcal{ML}(\mathbb{H}^2)$ is the total lift of M . Then β is C^1 -smooth on each component of $\mathbb{H}^2 \setminus \tilde{M}$. Therefore, there is a regular neighborhood of β stated as in the following. Since $\pi_1(S)$ acts on the universal cover of τ , \mathbb{H}^2 , as deck transformations, $\pi_1(S)$ naturally acts on $\mathbb{H}^2 \times (-1, 1)$ by $\gamma \cdot (x, t) = (\gamma \cdot x, t)$ for all $\gamma \in \pi_1(S)$, $x \in \mathbb{H}^2$ and $t \in (-1, 1)$. Then, letting $\zeta: \mathbb{H}^2 \rightarrow \mathbb{H}^2 \times (-1, 1)$ be the canonical embedding onto $\mathbb{H}^2 \times \{0\}$ defined by $x \mapsto (x, 0)$, we have

Lemma 5.8 (regular neighborhood). *There is an immersion $\iota: \mathbb{H}^2 \times (-1, 1) \rightarrow \mathbb{H}^3$ such that:*

- (i) $\beta = \iota \circ \zeta$,
- (ii) ι is ρ -equivariant,

- (iii) for every $x \in \mathbb{H}^2$, there exists a neighborhood U of x in \mathbb{H}^2 such that $\iota|U \times (-1, 1)$ is an embedding, and
- (iv) for the closure P of each component of $\mathbb{H}^2 \setminus \tilde{M}$, ι is C^1 -smooth on $P \times (-1, 1)$.

Then, since S is closed, (iii) immediately implies that

- (v) If $\epsilon > 0$ is sufficiently small, then, for every $x \in \mathbb{H}^2$, there is a unique topological 3-ball $B_x \subset \mathbb{H}^2 \times (-1, 1)$ containing $x \times \{0\}$ such that ι embeds B_x onto the round ball of radius ϵ centered at $\iota((x, 0)) = \beta(x)$ in \mathbb{H}^3 and B_x changes continuously in $x \in \mathbb{H}^2$.

Consider the open 1-neighborhood of a totally geodesic hyperplane in \mathbb{H}^3 . Then this 1-neighborhood has a canonical product structure, $\mathbb{H}^2 \times (-1, 1)$, whose coordinates are given by the nearest point projection onto the hyperplane and the distance from the hyperplane. Then we canonically identify this 1-neighborhood with the domain of ι in Lemma 5.8. Let $N = \mathbb{H}^2 \times (-1, 1)$. Then we equip N with the pullback metric of \mathbb{H}^3 via ι , unless otherwise stated. However, we may also regard N as the 1-neighborhood contained in \mathbb{H}^3 and equip N with the metric induced from the ambient space, so that N is convex.

Recall that we have a sequence of bending maps $\beta_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ ($i \in \mathbb{N}$). By Proposition 5.6, there is a homeomorphism $\eta_i: \tau \rightarrow \tau_i$ for each $i \in \mathbb{N}$, such that $\beta_i \circ \tilde{\eta}_i$ converges to β uniformly in C^0 topology as $i \rightarrow \infty$. Then $\beta_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ naturally factors through a continuous map into N :

Proposition 5.9. *For every $\delta > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then there exists a unique continuous map $\zeta_i: \mathbb{H}^2 \rightarrow N$ such that*

- (i) $\beta_i = \iota \circ \zeta_i$,
- (ii) the action of $\pi_1(S)$ commutes with ζ_i ,
- (iii) $\zeta_i \circ \tilde{\eta}_i(x) \in B_x$ for all $x \in \mathbb{H}^2$, and
- (iv) $\zeta_i \circ \tilde{\eta}_i(x)$ and $\zeta(x)$ are δ -close for all $x \in \mathbb{H}^2$.

Proof. By Proposition 5.6, for every $\delta > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, such that β and $\beta_i \circ \tilde{\eta}_i$ are δ -close pointwise. Thus there exists a unique homotopy $\vartheta: [0, 1] \times \mathbb{H}^2 \rightarrow \mathbb{H}^3$ such that, letting $\vartheta(t, x) = \vartheta_t(x)$,

- $\vartheta_0 = \beta$,
- $\vartheta_1 = \beta_i \circ \tilde{\eta}_i$, and
- for each $x \in \mathbb{H}^2$, $\vartheta|[0, 1] \times \{x\}$ is the geodesic segment in \mathbb{H}^3 , parametrized by arc length, that connects $\beta(x)$ to $\beta_i \circ \tilde{\eta}_i(x)$.

Then ϑ is ρ -equivariant and the geodesic segment $\vartheta|[0, 1] \times \{x\}$ has length less than δ for all $x \in \mathbb{H}^2$.

Recall that, by Lemma 5.8 (v), for each point $x \in \mathbb{H}^2$, there exists a ball B_x in N such that: B_x contains $\zeta(x)$; $B_x \subset N$ changes continuously in x ; the immersion ι embeds B_x isometrically onto a round ball of radius (fixed) ϵ centered at $\beta(x)$. Therefore, since we can assume $\delta < \epsilon$, the homotopy ϑ uniquely factors through to a homotopy $\xi: [0, 1] \times \mathbb{H}^2 \rightarrow N$ so that, $\xi_0 = \zeta$ and $\iota \circ \xi = \vartheta$. Then, for all $x \in \mathbb{H}^2$, $\xi([0, 1] \times \{x\})$ is a geodesic segment of length less than δ starting at $\zeta(x)$ and, thus, contained in B_x .

Since $\iota \circ \xi = \vartheta$, we have $\iota \circ \xi_1 = \beta_i \circ \tilde{\eta}_i$. Then let $\zeta_i = \xi_1 \circ \tilde{\eta}_i^{-1}: \mathbb{H}^2 \rightarrow N$. Then $\iota \circ \zeta_i = \beta_i$; thus (i) holds. Besides, for all $x \in \mathbb{H}^2$, $\zeta_i \circ \tilde{\eta}_i(x) = \xi(1, x) \in B_x$; thus (iii) holds. Since ϑ and ι are ρ -equivariant and $B_{\gamma \cdot x} = \gamma B_x$ for all $x \in \mathbb{H}^2$ and $\gamma \in \pi_1(S)$, therefore ζ_i commutes with the action of $\pi_1(S)$; thus (ii) holds. In addition, we have $\zeta(x) = \xi_0(x)$ and $\zeta_i \circ \tilde{\eta}_i(x) = \xi_1(x)$. Then, since the geodesic segment $\xi([0, 1] \times \{x\})$ has length less than δ , its endpoints $\zeta_i \circ \tilde{\eta}_i(x)$ and $\zeta(x)$ have length less than δ for all $x \in \mathbb{H}^2$; thus (iv) holds. \square

Case 2. Suppose that L contains a closed leaf with weight π modulo 2π . Let M be the multiloop on τ consisting of all such closed leaves of L . In this case, for the closure P of each components of $\mathbb{H}^2 \setminus \tilde{M}$, we will similarly have an open regular neighborhood $\iota_P: N_P \rightarrow \mathbb{H}^3$ of $\beta|P$.

Regard P as a convex subset contained in a totally geodesic hyperplane in \mathbb{H}^3 , and take an open ϵ -neighborhood N_P of P in \mathbb{H}^3 . Then let $\zeta_P: P \rightarrow N_P \subset \mathbb{H}^3$ denote the canonical embedding. We let $Stab(P)$ denote the maximal subgroup of $\pi_1(S)$ that preserves P . Then $Stab(P)$ isometrically acts on N_P . The nearest point projection of \mathbb{H}^3 to P induces a projection $\Phi_P: N_P \rightarrow P$. Then $N_P \setminus P$ has a natural product structure $\partial N_P \times (0, 1)$: Namely, we associate each point $x \in N_P \setminus P$ with the point $y \in \partial N_P$ such that the geodesic segment from y to $\Phi_P(y)$ contains x and with the distance $dist_{\mathbb{H}^3}(x, P) \in (0, \epsilon)$ times $1/\epsilon$. Thus we naturally set $N_P \cong \partial N_P \times [0, 1]/\sim$, where $(x, t_x) \sim (y, t_y)$ if $t_x = t_y = 0$ and $\Phi_P(x) = \Phi_P(y)$. This product structure is preserved under action of $Stab(P)$. Then, since $\beta|P$ is locally injective, similarly to Case 1, we have

Lemma 5.10 (regular neighborhood). *There exists an immersion $\iota_P: N_P \rightarrow \mathbb{H}^3$ such that*

- (i) $\iota_P \circ \zeta_P = \beta|P$,
- (ii) ι_P is $\rho|Stab(P)$ -equivariant,
- (iii) for every $x \in P$, there exists a neighborhood U of x in P such that $\Phi_P^{-1}(U)$ embeds into \mathbb{H}^3 via ι_P , and

- (iv) for each component of P minus the periodic leaves of \tilde{L} contained in $\text{int}(P)$, letting Q be its closure, ι_P is C^1 -smooth on $\Phi^{-1}(Q)$.

Then, since $P/\text{Stab}(P)$ is compact, (ii) and (iii) implies

- (v) there exists $\epsilon > 0$ such that, for each point $x \in P$, there exists a (unique) ball $B_x \subset N_P$ containing x such that ι_P embeds B_x onto a round ball of radius ϵ centered at $\iota_P(x) = \beta(x)$ and that B_x changes continuously in $x \in P$.

Recalling that P is a convex subset of \mathbb{H}^2 , for a boundary component l of P , let $N_\epsilon(l, P)$ denote the ϵ -neighborhood of l in P . Then let $R_\epsilon(l) = \Phi_P^{-1}(N_\epsilon(l, P)) \subset N_P$.

We equip N_P with the pull back metric of \mathbb{H}^3 via ι_P , unless otherwise stated. However we have first defined N_P as a convex subset of \mathbb{H}^3 and, as in Case 1, we may use the metric induced by this inclusion when stated so. Indeed,

Lemma 5.11. *If $\epsilon > 0$ is sufficiently small, then, in addition to (i)–(v) in Lemma 5.10, we have*

- (vi) if l and l' are different boundary components of P , then $R_\epsilon(l)$ and $R_\epsilon(l')$ are disjoint subsets of N_P , and
- (vii) for each boundary component l of P , the two metrics on $R_\epsilon(l)$ coincide and $\Phi_P|_{R_\epsilon(l)}$ is the nearest point projection onto $N_\epsilon(l, P)$ with respect to the pullback metric via ι_P as well.

Proof. Since the action of $\text{Stab}(P)$ on P is cocompact, there is a positive lower bound for the distance between any distinct boundary geodesics of $P \subset \mathbb{H}^2$. Thus, we have (vi).

Each periodic leaf of L is isolated. Thus, for each boundary component l of P , if $\epsilon > 0$ is sufficiently small, then ι_P isometrically embeds $N_\epsilon(l, P)$ into a totally geodesic hyperplane in \mathbb{H}^3 . Thus we have (vii). \square

Let l be a leaf of \tilde{M} ; then l bounds exactly two components of $\mathbb{H}^2 \setminus \tilde{M}$. Let P_1 and P_2 be the closures of those components. By Lemma 5.11 (vii), for each $j = 1, 2$, $N_\epsilon(l, P_j) \subset P_j$ isometrically embeds into N_{P_j} via ζ_{P_j} ; then the image $\zeta_{P_j}(N_\epsilon(l, P_j))$ isometrically embeds, via ι_{P_j} , into a totally geodesic hyperplane in \mathbb{H}^3 . Moreover, since the weight of l is π modulo 2π , we have $\iota_{P_1} \circ \zeta_{P_1}(N_\epsilon(l, P_1)) = \iota_{P_2} \circ \zeta_{P_2}(N_\epsilon(l, P_2))$. Thus, by Lemma 5.11 (vii), there is a unique isometry $\tilde{\psi}_l$ from $R_\epsilon(\zeta_{P_1}(l)) =: R_1 \subset N_{P_1}$ onto $R_\epsilon(\zeta_{P_2}(l)) =: R_2 \subset N_{P_2}$ taking $N_\epsilon(l_1, P_1)$ to $N_\epsilon(l_2, P_2)$, so that $\iota_{P_2} \circ \tilde{\psi}_l = \iota_{P_1}$ on R_1 . Thus we can quotient $P_1 \cup P_2 \subset \mathbb{H}^2$ by identifying $N_\epsilon(l, P_1)$ and $N_\epsilon(l, P_2)$ via $\tilde{\psi}_l$. Let $(P_1 \cup P_2)/\tilde{\psi}_l$ denote the

quotient, and let $\tilde{\psi}_l: P_1 \cup P_2 \rightarrow (P_1 \cup P_2)/\tilde{\psi}_l$ also denote this quotient map. Note this quotient $(P_1 \cup P_2)/\tilde{\psi}_l$ is exactly the branched surface obtained from the convex subset $P_1 \cup P_2 \subset \mathbb{H}^2$ by quotienting the ϵ -neighborhood of l , $N_\epsilon(l, P_1 \cup P_2)$, by the isometric reflection fixing l . Let $m \subset \tau$ be the loop of M that lifts to $l \subset \mathbb{H}^2$. Then its ϵ -neighborhood $N_\epsilon(m)$ in τ is a cylinder, and the isometric reflection of $N_\epsilon(l, P_1 \cup P_2)$ about l descends to the isometric reflection of $N_\epsilon(m)$ about m .

Let \mathcal{R}_l denote $\iota_{P_1}(R_1) = \iota_{P_2}(R_2) \subset \mathbb{H}^3$. Then, clearly,

Lemma 5.12. *Both projections $\Phi_{P_1}: R_1 \rightarrow N_\epsilon(l, P_1)$ and $\Phi_{P_2}: R_2 \rightarrow N_\epsilon(l, P_2)$, via ι_{P_1} and ι_{P_2} , descend to the same nearest point projection*

$$\Phi_l: \mathcal{R}_l \rightarrow \iota_{P_1}(N_\epsilon(l, P_1)) = \iota_{P_2}(N_\epsilon(l, P_2))$$

in \mathbb{H}^3 , so that, by $\iota_{P_j}(N_\epsilon(l, P_j)) \cong N_\epsilon(l, P_j)/\tilde{\psi}_l \subset (P_1 \cup P_2)/\tilde{\psi}_l$, we have $\Phi_l \circ \iota_{P_j} = \psi_l \circ \Phi_{P_j}$ on R_j for both $j = 1, 2$.

Consider the ϵ -neighborhoods $N_\epsilon(m)$ of all loops m of M ; then they are disjoint by Lemma 5.11 (vi). Let $\psi_\epsilon: \tau \rightarrow \tau_\epsilon$ denote the quotient map that simultaneously quotients $N_\epsilon(m)$ by the reflection about m for all loops m of M . Then τ_ϵ is a branched hyperbolic surface with the boundary $\psi_\epsilon(M)$ whose branched locus is the multiloop consisting of the ψ_ϵ -image of $\partial N_\epsilon(m)$ for all loops m of M .

Regard \mathbb{H}^2 as the union of the closures P of all components of $\mathbb{H}^2 \setminus \tilde{M}$. We can quotient $\mathbb{H}^2 = \cup P$ by quotienting the ϵ -neighborhoods of l by $\tilde{\psi}_l$ for all leaves l of \tilde{M} . Then this quotient is exactly the universal cover of τ_ϵ and the quotienting map descends $\pi_1(S)$ -action on \mathbb{H}^2 to the deck transformation of $\tilde{\tau}_\epsilon$. Observe that $\psi_\epsilon: \tau \rightarrow \tau_\epsilon$ is a quasiisometry. Then ψ_ϵ converges to an isometry as $\epsilon \searrow 0$. Thus we have

Lemma 5.13. *For every $\epsilon' > 0$, there exists $E > 0$, such that, if $E > \epsilon > 0$, then*

$$(1 - \epsilon') \cdot \text{length}_\tau(l) < \text{length}_{\tau_\epsilon}(l')$$

for all geodesic loops l on τ and l' on τ_ϵ such that $\psi_\epsilon(l)$ is homotopic to l' .

Recall that we have the sequence of the bending maps $\beta_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ ($i \in \mathbb{N}$) associated with $C_i \in \mathcal{P}_\rho$ and the marking-preserving homeomorphism $\eta_i: \tau \rightarrow \tau_i$ given by Proposition 5.6 such that $\beta_i \circ \tilde{\eta}_i$ converging to β uniformly in C^0 -topology as $i \rightarrow \infty$. Let $M_i = \eta_i(M)$, a multiloop on τ_i , and let $\tilde{M}_i \subset \mathbb{H}^2$ be the total lift of M_i . Then we have

Proposition 5.14. *For every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large, then, for the closures Q and P of corresponding components of $\mathbb{H}^2 \setminus \tilde{M}_i$ and*

$\mathbb{H}^2 \setminus \tilde{M}$, respectively, there is a unique continuous map $\zeta_{i,P}: Q \rightarrow N_P$ such that:

- (i) $\beta_i = \iota_P \circ \zeta_{i,P}$ on Q and,
- (ii) the action of $\text{Stab}(Q) \cong \text{Stab}(P)$ commutes with $\zeta_{i,P}$,
- (iii) $\zeta_{i,P} \circ \tilde{\eta}_i(x) \in B_x$ for all $x \in P$,
- (iv) $\zeta_{i,P} \circ \tilde{\eta}_i(x)$ and $\zeta(x)$ are δ -close for all $x \in P$.

The proof of this proposition is similar to the proof of Proposition 5.9.

5.4. Convergence of hyperbolic structures. The following proposition is the main step to prove Theorem 5.4.

Proposition 5.15. $\tau_i \rightarrow \tau$ as $i \rightarrow \infty$.

Proposition 5.15 follows immediately from Proposition 5.16 and Theorem 5.17 below.

Proposition 5.16. For every $\epsilon > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then $\text{length}_\tau(l) < (1 + \epsilon) \cdot \text{length}_{\tau_i}(l_i)$ for all pairs of geodesic loops l and l_i on τ and τ_i , respectively, that are homotopic (regarded as loops on S).

Theorem 5.17. ([18]) Consider the function $K: \text{Teich}(S) \times \text{Teich}(S) \rightarrow \mathbb{R}$ given by

$$K(\tau, \tau') = \sup \left(\ln \left(\frac{\text{length}_\tau(l)}{\text{length}_{\tau'}(l)} \right) \right),$$

where the supremum runs over all pairs of geodesic loops l and l' on τ and τ' , that are homotopic (regarded loops on S). Then K is well-defined and it defines an antisymmetric distance on $\text{Teich}(S)$.

Proof of Proposition 5.15 assuming Proposition 5.16. By Proposition 5.16, $K(\tau_i, \tau) \rightarrow 0$ as $i \rightarrow \infty$. Therefore, by Theorem 5.17, $\tau_i \rightarrow \tau$ as $i \rightarrow \infty$. \square

Proof of Proposition 5.16. Case 1. First suppose that L contains no closed leaf of weight π modulo 2π . Then we have, by Lemma 5.8, a piecewise C^1 -smooth regular neighborhood $\iota: N \rightarrow \mathbb{H}^3$ of β where $N \cong \mathbb{H}^2 \times (-1, 1)$, and β factors through $\zeta: \mathbb{H}^2 \rightarrow N$, namely $\beta = \iota \circ \zeta$. Recall that N is equipped with the pullback metric of \mathbb{H}^3 under ι .

Let

$$\Phi: N \cong \mathbb{H}^2 \times (-1, 1) \rightarrow \mathbb{H}^2 \times \{0\}$$

denote the canonical projection, i.e. $(x, t) \mapsto (x, 0)$. Then, since N is piecewise C^1 -smooth (Lemma 5.8 (vi)), Φ induces the map between the tangent space $T(N)$ of N to the tangent space of \mathbb{H}^2 in N :

$$\Phi^*: T(N) \rightarrow T_{\mathbb{H}^2 \times \{0\}}(N).$$

Then, since Φ fixes the image of $\zeta: \mathbb{H}^2 \rightarrow N$, by identifying $T_{\mathbb{H}^2 \times \{0\}}(N)$ with $T(\mathbb{H}^2)$, we have

Lemma 5.18. *For every $\epsilon > 0$, if $\delta \in (0, 1]$ is sufficiently small, then*

$$\Phi^*|_{T_{(x,t)}N}: T_{(x,t)}N \rightarrow T_x\mathbb{H}^2$$

is a $(1 + \epsilon)$ -lipschitz map for every $(x, t) \in \mathbb{H}^2 \times (-\delta, \delta) \subset N$.

Fix (small) $\epsilon > 0$ and then fix $\delta > 0$ obtained by Lemma 5.18. Let $\zeta_i: \mathbb{H}^2 \rightarrow N$ be denote the continuous map obtained by Proposition 5.9 for sufficiently large $i \in \mathbb{N}$. Then, by Proposition 5.9 (iv), ζ_i maps into $\mathbb{H}^2 \times (-\delta, \delta)$, for sufficiently large i . Let l_i be a geodesic loop on τ_i . Regarding l_i as a simple closed curve on τ , let $p_i: [0, 1] \rightarrow \mathbb{H}^2$ be a path obtained by lifting l_i to the universal cover of τ_i , so that $\text{length}_{\tau_i}(l_i) = \text{length}_{\mathbb{H}^2}(p_i)$. Let α denote the element of $\pi_1(S)$ that identifies the endpoints of p_i . Let $q_i = \zeta_i \circ p_i: [0, 1] \rightarrow N$. Then $\text{length}_{\mathbb{H}^2}(p_i) = \text{length}_N(q_i)$, since β_i is a local isometry onto its image, in \mathbb{H}^3 , with intrinsic metric. Besides, if $i \in \mathbb{N}$ is sufficiently large, the path q_i is contained in $\mathbb{H}^2 \times (-\delta, \delta)$, for all geodesic loops l_i on τ_i . Let

$$q'_i = \Phi \circ q_i: [0, 1] \rightarrow \text{Im } \zeta \cong \mathbb{H}^2.$$

Then, by Lemma 5.18, we have

$$\text{length}(q'_i) \leq (1 + \epsilon) \cdot \text{length}(q_i) = (1 + \epsilon) \cdot \text{length}(l_i).$$

Since α identifies the endpoints of p_i and the action of $\pi_1(S)$ commutes with both ζ_i (Proposition 5.9 (ii)) and Φ , therefore α identifies the endpoints of q_i and, thus, the endpoints of q'_i . Then q'_i descends to a closed curve on τ that is homotopic to l_i . Let l be the geodesic representative of the close curve on τ . Then $\text{length}_{\tau}(l) \leq \text{length}_{\mathbb{H}^2}(q'_i)$. Therefore, we have

$$\text{length}(l) \leq (1 + \epsilon) \cdot \text{length}(l_i)$$

for sufficiently large $i \in \mathbb{N}$.

Case 2. Suppose that L contains a close leaf of weight π modulo 2π . Let M be the sublamination of L consisting of all such periodic leaves. For each component $\mathbb{H}^2 \setminus \tilde{M}$, letting P be its closure, we have constructed a regular ϵ -neighborhood of $\beta|P$ (Lemma 5.10 and Lemma 5.11). Then, recalling $N_P = \partial N_P \times [0, 1] / \sim$, for $\delta \in (0, 1]$, let $N_{P,\delta} = \partial N_P \times [0, \delta] / \sim$. Then we can easily see an analogue of Lemma 5.18:

Lemma 5.19. *For every $\epsilon > 0$, there exists $\delta > 0$, such that, if P is the closure of a component of $\mathbb{H}^2 \setminus \tilde{M}$, then, for all $z \in N_{P,\delta}$, the piecewise C^1 -smooth projection $\Phi_P: N_P \rightarrow N_P$ induces a $(1 + \epsilon)$ -lipschitz map*

from the tangent space of N_P at z to the tangent space of P at the point $x \in P$ such that $\zeta_P(x) = \Phi_P(z)$:

$$\Phi_P^*|T_z N_P: T_z N_P \rightarrow T_x P.$$

Given a regular neighborhood of $\beta|P$, since β is ρ -equivariant, we can easily make a regular neighborhood of $\beta|\gamma P$ for all $\gamma \in \pi_1(S)$. Namely we set

$$\begin{aligned} N_{\gamma P} &:= N_P, \\ \Phi_{\gamma P} &:= \Phi_P \\ \iota_{\gamma P} &:= \rho(\gamma) \cdot \iota_P: N_{\gamma P} \rightarrow \mathbb{H}^3 \\ \zeta_{\gamma P} &:= \zeta_P \cdot \gamma^{-1}: \gamma P \rightarrow N_P, \end{aligned}$$

which satisfy (i) - (vii) in Lemma 5.10 and Lemma 5.11. Therefore we can assume

Claim 5.20. *The regular ϵ -neighborhoods of $\beta|P$ and $\beta|\gamma P$ satisfy the above relations for every $\gamma \in \pi_1(S)$ and the closure P of every component of $\mathbb{H}^2 \setminus \tilde{M}$.*

Let m be a geodesic loop on τ_i . Then pick a partition of m into finitely many disjoint arcs $m_j (j \in \mathbb{Z}_n)$ with some $n \in \mathbb{N}$ indexed in the cyclic order, such that, for each $j \in \mathbb{Z}_n$, $\eta_i^{-1}(m_j)$ is contained in the closure of a component of $\tau_i \setminus M$. Let \tilde{m}_j be a lift of m_j to the universal cover, \mathbb{H}^2 , of τ_i . Then $\tilde{\eta}_i^{-1}(\tilde{m}_j)$ is contained in the closure P_j of a component of $\mathbb{H}^2 \setminus \tilde{M}$. By claim 5.20, we may change the choice of the lift \tilde{m}_j by an element of $\pi_1(S)$ in the following argument, if necessarily.

Let $Q_j = \tilde{\eta}_i(P_j)$ and let $\zeta_{i,P_j}: Q_j \rightarrow N_{P_j}$ be the continuous map obtained by Proposition 5.14. Then we have $\iota_{P_j} \circ \zeta_{i,P_j}|_{\tilde{m}_j} = \beta_i|_{\tilde{m}_j}$. Then, since β_i preserves length of a curve in its domain, so does ζ_{i,P_j} . In particular, ζ_{i,P_j} preserves the length of \tilde{m}_j . Let q_j denote the curve $\Phi_{P_j} \circ \zeta_{i,P_j}|_{\tilde{m}_j}$ in $\text{Im } \zeta_{P_i} \cong P_j$.

Proposition 5.21. *For every $\epsilon > 0$, there exists $I > 0$, such that, if $i > I$ then, for every geodesic loop m on τ_i and every finite partition of m into segments m_1, m_2, \dots, m_n as above, we have*

$$(1 + \epsilon) \cdot \text{length}_{\tau_i}(m_j) \geq \text{length}_{P_j}(q_j)$$

for each $j \in \{1, 2, \dots, n\}$.

Proof. By Proposition 5.14, for every $\delta > 0$, if $I \in \mathbb{N}$ is sufficiently large, then $\zeta_{i,P_j}|_{\tilde{m}_j}$ is contained in $N_{P_j,\delta}$ for each $j \in \{1, \dots, n\}$. Then, similarly to Case 1, Lemma 5.19 implies the proposition. \square

Since m_j and m_{j+1} share an endpoint, accordingly, an endpoint of q_j corresponds to an endpoint q_{j+1} . Let $\Psi: \mathbb{H}^2 \rightarrow \tau$ be the universal covering map. Then, Ψ take the corresponding endpoints of $q_j \subset P_j$ and $q_{j+1} \subset P_{j+1}$ typically to different points on τ . However, recalling the quasiisometry $\psi_\epsilon: \tau \rightarrow \tau_\epsilon$ from §5.3, we show

Lemma 5.22. *The corresponding endpoints of q_j and q_{j+1} map to the same points under $\psi_\epsilon \circ \Psi$ for each $j \in \mathbb{Z}_n$; therefore $\cup_j q_j \subset \mathbb{H}^2$ descends to a closed curve on τ_ϵ . Furthermore this closed curve is homotopic to $\psi_\epsilon \circ \tilde{\eta}_i^{-1}(m)$.*

Proof. Consider consecutive segments m_j and m_{j+1} , sharing an endpoint v . Then we can assume that \tilde{m}_j and \tilde{m}_{j+1} share an endpoint \tilde{v} that is a lift of v to \mathbb{H}^2 (by an element of $\pi_1(S)$). Thus, to show that $\psi_\epsilon \circ \Psi(q_j)$ and $\psi_\epsilon \circ \Psi(q_{j+1})$ share an endpoint corresponding to v , it suffices to show that $\tilde{\psi}_\epsilon \circ \Phi_{P_j} \circ \zeta_{P_j}(\tilde{v}) = \tilde{\psi}_\epsilon \circ \Phi_{P_{j+1}} \circ \zeta_{P_{j+1}}(\tilde{v}) \in \tilde{\tau}_\epsilon$, where $\tilde{\psi}_\epsilon: \tilde{\tau} \rightarrow \tilde{\tau}_\epsilon$ is the lift of $\psi_\epsilon: \tau \rightarrow \tau_\epsilon$ to the map between universal covers, \mathbb{H}^2 .

Since \tilde{m}_j and \tilde{m}_{j+1} share an endpoint, either $P_j = P_{j+1}$ or P_j and P_{j+1} are the closures of adjacent components of $\mathbb{H}^2 \setminus \tilde{M}$. Suppose that $P_j = P_{j+1} (= P)$. Then, letting $Q = \tilde{\eta}_i(P)$, the segments \tilde{m}_j and \tilde{m}_{j+1} are contained in Q , sharing an endpoint \tilde{v} . Thus the claim is clear.

Next suppose that P_j and P_{j+1} are adjacent. Then let l be the common boundary geodesic of P_j and P_{j+1} , which is a leaf of \tilde{M} . Then $\tilde{\eta}_i^{-1}(\tilde{v}) \in l$. As in Case 2 of §5.3, for each $k = j, j+1$, let $R_{k,\epsilon}(l) \subset N_{P_k}$ be $\Phi_{P_k}^{-1}(N_\epsilon(l, P_k))$, where $N_\epsilon(l, P_k)$ is the ϵ -neighborhood of l in P_k with $\epsilon > 0$ given by Lemma 5.11. Then we have a canonical isometry $\tilde{\psi}_l: R_{k,\epsilon}(l_j) \rightarrow R_{k,\epsilon}(l_{j+1})$, which coincides with the reflection $\tilde{\psi}_\epsilon|_{P_j \cup P_{j+1}}$. Since $\tilde{\eta}_i^{-1}(\tilde{v}) \in l$, by Proposition 5.14 (iv), assuming that $i \in \mathbb{N}$ is sufficiently large, then $\zeta_{P_k}(\tilde{v})$ is contained in $R_{k,\epsilon}(l_k)$ for each $k = j, j+1$. Then $\tilde{\psi}_l$ identifies $\zeta_{P_j}(\tilde{v})$ and $\zeta_{P_{j+1}}(\tilde{v})$ since $\iota_{P_j} = \iota_{P_{j+1}} \circ \tilde{\psi}_l$ on $R_{k,\epsilon}(l_j)$. Therefore, by Lemma 5.12, we have $\tilde{\psi}_\epsilon \circ \Phi_{P_{j+1}} \circ \zeta_{P_{j+1}}(\tilde{v}) = \tilde{\psi}_\epsilon \circ \Phi_{P_j} \circ \zeta_{P_j}(\tilde{v}) \in \tilde{\tau}_\epsilon$ as desired. Thus $\cup_j q_j$ maps to a loop q via $\psi_\epsilon \circ \Psi$.

We next show that $\psi_\epsilon \circ \eta_i^{-1}(m)$ is homotopic to the loop q on τ_ϵ . Consider a lift \tilde{m} ($\cong \mathbb{R}$) of m to the universal cover of τ_i , and let $[m]$ be the element of $\pi_1(S)$ that preserves \tilde{m} . Then the partition of m into m_1, \dots, m_n lifts an infinite partition of \tilde{m} that is invariant under the action of $[m] \in \pi_1(S)$. Each segment of this partition of \tilde{m} is a lift \tilde{m}_j of the segment m_j with some $j \in \mathbb{Z}_n$, and $\Psi_{P_j} \circ \zeta_{P_j}$ takes \tilde{m}_j into $P_j \subset \tilde{\tau}$. Then, since $\eta_i: \tau \rightarrow \tau_i$ is a marking-preserving homeomorphism, the union of $\Psi_{P_j} \circ \iota_{P_j}(\tilde{m}_j)$ over all such segments \tilde{m}_j of \tilde{m} is a quasigeodesic in $\tilde{\tau}$ invariant under $[m]$, which corresponds to

the loop $\tilde{\eta}_i^{-1}(m)$ on τ . Then $\tilde{\psi}_\epsilon: \tilde{\tau} \rightarrow \tilde{\tau}_\epsilon$ takes this quasigeodesic to a (continuous) quasigeodesic that projects to q via the universal covering map. Therefore $\psi_\epsilon \circ \eta_i^{-1}(m)$ is homotopic to q . \square

Claim 5.23. *For every $\epsilon > 0$, there exists $I > 0$, such that if $i > I$, then*

$$(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(l)$$

for all geodesic loops m and l on τ_i and τ_ϵ , respectively, such that $\psi_\epsilon \circ \eta_i^{-1}(m)$ is homotopic to l .

Proof. By Proposition 5.21, for every $\epsilon > 0$, there exists $I \in \mathbb{N}$ such that, if $i > I$, given any geodesic loop m on τ_i and a finite partition $m = m_1 \cup m_2 \cup \dots \cup m_n$ as above, such that $(1 + \epsilon) \cdot \text{length}_{\tau_i}(m_j) \geq \text{length}_{\tilde{\tau}}(q_j)$ for each $j \in \mathbb{Z}_n$, where q_j are the geodesic segments on $\tilde{\tau}$ defined as above. Then

$$\begin{aligned} (1 + \epsilon) \cdot \text{length}_{\tau_i}(m) &= (1 + \epsilon) \cdot \cup_j \text{length}_{\tau_i}(m_j) \\ &\geq \sum_j \text{length}_{\tilde{\tau}}(q_j) \\ &= (\sum_j \text{length}_{\tilde{\tau}}(q_j)). \end{aligned}$$

By Lemma 5.22, $\cup_j q_j \subset \tilde{\tau}$ descends to a loop on τ_ϵ under $\psi_l \circ \Psi$. Thus we have

$$(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(q).$$

Since l is the shortest loop homotopic to q on τ_ϵ . Hence

$$(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(l).$$

\square

Fix $\epsilon' > 0$. Then pick sufficiently small $\epsilon > 0$ and sufficiently large $i \in \mathbb{N}$ so that they satisfy Lemma 5.13 and Proposition 5.23. Then for all geodesic loops l on τ , l_i on τ_i and l_ϵ on τ_ϵ such that $\psi_\epsilon(l)$ is homotopic to l_ϵ and $\eta_i(l)$ is homotopic to l_i , we have

$$(1 - \epsilon') \cdot \text{length}_\tau(l) < \text{length}_{\tau_\epsilon}(l_\epsilon) < (1 + \epsilon') \cdot \text{length}_{\tau_i}(l_i)$$

This completes the proof of Case 2. 5.16

5.5. Convergence of bending maps: the proof of Theorem 5.4.

Recall that we have been assuming the λ_i converges to the geodesic lamination λ_∞ in $\mathcal{GL}(S)$ as $i \rightarrow \infty$. Then, in Proposition 5.15, we have shown that $\tau_i \rightarrow \tau \in \text{Teich}(S)$. Thus we let $\eta_i: \tau \rightarrow \tau_i$ be a marking-preserving diffeomorphism for each $i \in \mathbb{N}$, so that η_i converges to the trivial isometry from τ to itself. Then, for every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then η_i is $(1 + \epsilon)$ -bilipschitz. Clearly (τ_i, λ_i) converges

to (τ, ν_∞) in $\text{Teich}(S) \times \mathcal{GL}(S)$. Let ϖ_i denote the geodesic lamination on τ_i that corresponds to $\nu_\infty \in \mathcal{GL}(\tau)$. Then $\angle_{\tau_i}(\varpi_i, \lambda_i) \rightarrow 0$ as $i \rightarrow \infty$. Let P be the closure of a component of $\mathbb{H}^2 \setminus \tilde{\nu}_\infty$, where $\tilde{\nu}_\infty \in \mathcal{GL}(\mathbb{H}^2)$ is the total lift of ν_∞ . Then let P_i be the closure of a component of $\mathbb{H}^2 \setminus \tilde{\varpi}_i$ corresponding to P , where $\tilde{\varpi}_i$ is the total lift of ϖ_i . Since $\eta_i: \tau \rightarrow \tau_i$ converges to the trivial isometry as $i \rightarrow \infty$, letting $\tilde{\eta}_i: \tilde{\tau} \rightarrow \tilde{\tau}_i$ denote the lift of η_i to an isometry between the universal covers of τ and τ_i , we clearly have

Lemma 5.24. *$\tilde{\eta}_i(P_i)$ converges to P uniformly.*

Moreover,

Proposition 5.25. *for every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\beta_i \circ \tilde{\eta}_i$ and β are ϵ -close pointwise in P . Furthermore $\beta_i \circ \tilde{\eta}_i$ converges to β in C^1 -topology uniformly in $P \setminus N_\epsilon(\partial P)$ as $i \rightarrow \infty$; then since η_i are smooth and orientation preserving, the normal vectors also converges in the interior of P .*

Proof. We first show

Lemma 5.26 (c.f. Lemma 5.7). *For every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\beta_i|_{P_i}: P_i \rightarrow \mathbb{H}^3$ is a $(1 + \epsilon, \epsilon)$ -quasiisometric embedding for the closure P_i of each component $\mathbb{H}^2 \setminus \tilde{\varpi}_i$.*

Proof. For a geodesic segment s on \mathbb{H}^2 , let s_ϵ denote s minus the $(\epsilon/2)$ -neighborhood of the endpoints of s , so that s_ϵ is a subsegment of s whose length is $\text{length}(s) - \epsilon$ (thus $s_\epsilon = \emptyset$, if $\text{length}(s) \leq \epsilon$). For every $\epsilon > 0$, if i is sufficiently large, then λ_i is contained the ϵ -neighborhood of ϖ_i in τ_i . Thus, if i is sufficiently, for all geodesic segments s in $\mathbb{H}^2 \setminus \tilde{\varpi}_i$, we have $\angle(s_\epsilon, \tilde{\lambda}_i) < \epsilon$, where $\tilde{\lambda}_i$ is the total lift of λ_i . Therefore, the lemma follows from Proposition 4.2 (i). \square

Corollary 5.27. *For every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\beta_i \circ \tilde{\eta}_i|_P$ is a $(1 + \epsilon, \epsilon)$ -quasiisometric embedding for the closure P of each component of $\mathbb{H}^2 \setminus \tilde{\nu}_\infty$.*

Proof. By Lemma 5.24, for every $\epsilon > 0$, if i is sufficiently large, then $\tilde{\eta}_i(P)$ is contained in the ϵ -neighborhood of P_i and, in addition, $\tilde{\eta}_i$ is $(1 + \epsilon)$ -bilipschitz. Thus Lemma 5.26 immediately implies the corollary. \square

Since $\beta|_P$ is an isometric embedding into a totally geodesic hyperplane in \mathbb{H}^3 , $\beta|_P: P \rightarrow \mathbb{H}^3$ continuously extends to a map from the ideal boundary $\partial_\infty P$ of P into $\hat{\mathbb{C}}$, where $\partial_\infty P$ is a subset of $\partial \mathbb{H}^2 \cong \mathbb{S}^1$. In addition, for sufficiently small $\epsilon > 0$, if i is sufficiently large, then, by Corollary 5.27, $\beta_i \circ \tilde{\eta}_i|_P$ is a $(1 + \epsilon, \epsilon)$ -quasiisometric embedding. Then

$\beta_i \circ \tilde{\eta}_i|P$ also extends to the ideal boundary $\partial_\infty P$ of P . Since $\beta_i \circ \tilde{\eta}_i$ and β are ρ -equivalently homotopic (Lemma 3.3), $\beta_i \circ \tilde{\eta}_i|_{\partial_\infty P} = \beta|_{\partial_\infty P}$. Since P is the closure of a component of $\mathbb{H}^2 \setminus \tilde{\nu}_\infty$, $\partial_\infty P$ contains at least 3 points. Therefore, Corollary 5.27 implies that $\beta_i \circ \eta$ and β are ϵ -close at each point in P .

Last we show the C^1 -convergence in the interior of P . For every $\epsilon > 0$, if i is sufficiently large, then $P_i \setminus N_\epsilon(\partial P_i)$ and $\tilde{\lambda}_i$ are disjoint. Thus β_i embeds $P_i \setminus N_\epsilon(\partial P_i)$ isometrically on to a totally geodesic hyperplane H_i in \mathbb{H}^3 . This hyperplane H_i converges to the hyperplane containing $\beta(P)$ as $i \rightarrow \infty$, since we have already shown the C^0 -convergence. Thus, since η_i is smooth and it converges to the trivial isometry, $\beta_i \circ \tilde{\eta}_i$ converges to β in C^1 -topology uniformly on $P \setminus N_\epsilon(\partial P)$. 5.25

Proof of Theorem 5.4. First assume that λ_i converges to $\lambda_\infty \in \mathcal{GL}(S)$ as $i \rightarrow \infty$ (as we have been assumed). Since there are only finitely many components of $\tau \setminus \nu_\infty$, therefore, by Proposition 5.25, $\beta_i \circ \tilde{\eta}_i$ converges to β uniformly in C^0 -topology on the closures of components of $\mathbb{H}^2 \setminus \tilde{\nu}_\infty$ and moreover, for every $\epsilon > 0$, in C^1 -topology in $\mathbb{H}^2 \setminus N_\epsilon(\tilde{\nu}_\infty)$. Then, since β_i are continuous and $\tilde{\nu}_\infty$ has empty interior, the C^0 -convergence extends to the entire domain \mathbb{H}^2 .

In general λ_i may *not* converge in $\mathcal{GL}(S)$. Thus suppose that $\beta_i \circ \tilde{\eta}_i$ does *not* converge to β in C^0 -topology with a sequence of any marking-preserving homeomorphisms $\eta_i: \tau \rightarrow \tau_i$. Then, taking a subsequence if necessarily, $\beta_i \circ \tilde{\eta}_i$ does *not* converge to β in C^0 -topology for any subsequence of $(\beta_i)_i$ and any sequence of marking-preserving homeomorphisms $\eta_i: \tau \rightarrow \tau_i$. Thus, since $\mathcal{GL}(S)$ is compact, we can in addition assume that λ_i converges in $\mathcal{GL}(S)$ as $i \rightarrow \infty$. This is a contradiction. 5.4

6. LOCAL CHARACTERIZATION OF PROJECTIVE STRUCTURES IN \mathcal{PML}

Recall that S is a closed orientable surface of genus at least 2 and projective structures have fixed orientation. The main theorem of this paper is:

Theorem 6.1. *Let $C = (\tau, L)$ be a projective structure on S with holonomy ρ . Then there is a neighborhood U of $[L]$ in $\mathcal{PML}(S)$ such that, for every projective structure $C' = (\tau', L') \in \mathcal{P}_\rho$ with $[L'] \in U$, then*

- (I) if $[L] \neq [L'] \in \mathcal{PML}(S)$, then $Gr_M(C) = C'$ for some admissible multiloop M on C and, moreover, M is a good approximation of $L' - L$, calculated on some traintrack T carrying both L and L' ,
- (II) if $[L] = [L'] \in \mathcal{PML}(S)$, then either
 - (i) $Gr_M(C) = C'$, where M is $L' - L$, or
 - (ii) $Gr_M(C') = C$, where M is $L - L'$, and
- (III) if $L = \emptyset$ or $L' = \emptyset$, then ρ is fuchsian and $C' = Gr_{L'}(C)$ or $C = Gr_L(C')$, respectively, and $U = \mathcal{PML}(S)$.

In (I), by “good approximation”, we mean that, for every $\epsilon > 0$, if U is sufficiently small, then, for each branch B of T , the weight of M on B is ϵ -close to the weight of L' minus the weight of L on B . In Case (II) if $L' - L > 0$ then (i) holds, if $L > L'$ then (ii) holds, and if $L = L'$ then $C = C'$.

Remark 6.2. We can regard (II) as the special case of (I), up to an exchange of C and C' . In (III), we may regard $[L] = [L']$ since $L = 0 \cdot L'$ or $L' = 0 \cdot L$; thus (III) is a special case of (II). Therefore (I) is essentially the main case. Note that (III) is Theorem 1.1.

The rest of this section is the proof of this theorem. We assume that projective surfaces are equipped with Thurston’s metric (§3.2.3), unless otherwise stated.

6.1. Projective structures on a quadrangle supported on a cylinder. Let \mathcal{A} be a round cylinder in $\hat{\mathbb{C}}$, that is, \mathcal{A} is bounded by disjoint round circles c_{-1} and c_1 . Let g denote the geodesic in \mathbb{H}^3 orthogonal to c_{-1} and c_1 , that is, g is orthogonal to the totally geodesic hyperplanes, in \mathbb{H}^3 , bounded by c_{-1} and c_1 . Then, there is a unique foliation $\mathcal{F}_{\mathcal{A}}$ of \mathcal{A} whose leaves are round circles $\{c_t\}_{t \in [-1,1]}$ orthogonal to g , which we call the canonical foliation on $\mathcal{F}_{\mathcal{A}}$.

Definition 6.3. Let $C = (f, \rho)$ be a projective structure on a simply connected surface F , so that ρ is trivial. Let e be a simple curve on C . Then we say that e is supported on the round cylinder \mathcal{A} if f embeds e properly into \mathcal{A} so that e transversally intersects all leaves c_t of $\mathcal{F}_{\mathcal{A}}$.

Let R be a quadrangle, and let e_1, e_2, e_3, e_4 denote the edges of R , cyclically indexed along $\partial R (\simeq \mathbb{S}^1)$. Then

Definition 6.4. A projective structure $C = (f, \rho)$ on R is supported on the round cylinder \mathcal{A} if

- (i) e_1 and e_3 immerse into c_{-1} and c_1 via f , respectively, and
- (ii) e_2 and e_4 are supported on \mathcal{A} .

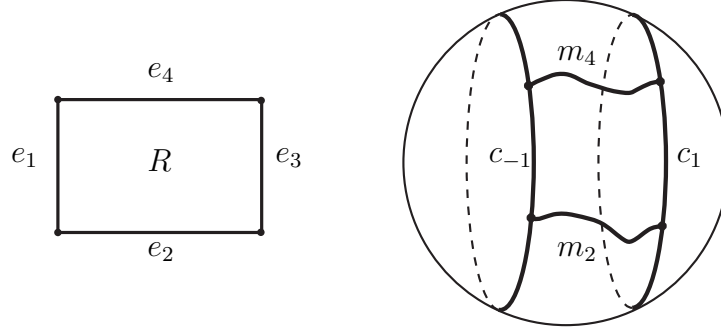


FIGURE 2.

Then the support of C is the round cylinder \mathcal{A} and the simple arcs $f(e_2)$ and $f(e_4)$ supported on \mathcal{A} . The canonical foliation $\mathcal{F}_{\mathcal{A}}$ on \mathcal{A} induces a foliation on the quadrangle C supported on \mathcal{A} .

6.2. Grafting a quadrangle supported on a cylinder. (c.f. [2, §3.5].) Let C be a projective structure on the quadrangle R supported on the round cylinder \mathcal{A} as above. Let m be a simple arc supported on \mathcal{A} . Then we can regard m also as an arc property embedded in \mathcal{A} . Thus, similarly to a grafting along a loop (§3.1), we can combine two projective structures C and \mathcal{A} , by cutting and pasting along m , and obtain a new projective structure on R supported on \mathcal{A} . Namely, there is a unique way to pair up and identify the boundary arcs of $C \setminus m$ and the boundary arcs of $\mathcal{A} \setminus m$ corresponding to m to make a connected projective surface. We call this operation the grafting of C along m and denote this resulting projective structure by $Gr_m(C)$. We call m an admissible arc on C (as for grafting along a loop). Then $Gr_m(C)$ is also an quadrangle, and the support of $Gr_m(C)$ is the same as that of C . If there is a multiarc M on C consisting of arcs supported on \mathcal{A} (admissible multiarc), then we can graft C along all arcs of M simultaneously and obtain a new projective structure on R with the same support. Then, we accordingly denote this resulting projective structure by $Gr_M(C)$.

Lemma 6.5. *Let C_1 and C_2 be projective structures on the quadrangle R that have the same support. Then, either $C_1 = Gr_M(C_2)$ or $C_2 = Gr_M(C_1)$ for some admissible multiarc M . Furthermore, the multiarc M is unique up to an isotopy of M on R through admissible multiarcs.*

Proof. Let \mathcal{A} be the round cylinder supporting C_1 and C_2 (as above). Let $f_1: R \rightarrow \mathcal{A}$ and $f_2: R \rightarrow \mathcal{A}$ be the developing maps of C_1 and C_2 , respectively. Let $\tilde{\mathcal{A}}$ be the universal cover of \mathcal{A} and $\Psi: \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ be the universal covering map. In addition, let m_2 and m_4 be the simple arcs property embedded in \mathcal{A} that support C_1 and C_2 , i.e, $m_2 = f_1(e_2) = f_2(e_2)$ and $m_4 = f_1(e_4) = f_2(e_4)$. Pick a lift \tilde{m}_4 of m_4 to $\tilde{\mathcal{A}}$. Then, for each $k = 1, 2$, $f_k: R \rightarrow \mathcal{A}$ uniquely lifts to $\tilde{f}_k: R \rightarrow \tilde{\mathcal{A}}$ so that $f_k = \Psi \circ \tilde{f}_k$ and \tilde{f}_k embeds e_4 onto \tilde{m}_4 . Clearly \tilde{f}_k is an embedding (although f_k may be *not*). Then $\tilde{f}_k(e_2)$ is a lift of m_2 to $\tilde{\mathcal{A}}$. Since projective structures have fixed orientation, $\tilde{f}_1(e_2)$ and $\tilde{f}_2(e_2)$ are in the same component of $\tilde{\mathcal{A}} \setminus \tilde{m}_4$. If $\tilde{f}_1(e_2) = \tilde{f}_2(e_2)$, then clearly $C_1 = C_2$. If $\tilde{f}_1(e_2) \neq \tilde{f}_2(e_2)$, without loss of generality, we can assume that $Im(\tilde{f}_2)$ is strictly contained in $Im(\tilde{f}_1)$, if necessarily, by exchanging C_1 and C_2 . Thus we can naturally regard $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$ as a projective structure on a quadrangle supported on \mathcal{A} , where its developing map is the restriction of Ψ to $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$. Then its supporting arc are both m_4 . Let d be the degree of the developing map of $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$ (at a point in $\mathcal{A} \setminus m_4$). We see that, by the elliptic isometries ϕ_t ($t \in \mathbb{S}^1$) of \mathbb{H}^3 fixing the geodesic g orthogonal to \mathcal{A} , we can foliate \mathcal{A} with the arcs $\phi_t(m_4)$.

Pick an admissible multiarc M on C_2 consisting of d disjoint arcs that embed, via f_2 , onto a leaf $\phi_t(m_4)$ of the foliation. Then we see that $C_1 = Gr_M(C_2)$, since the union of the projective structures inserted to C_2 by Gr_M is exactly $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$.

Let M' be an another admissible multiarc on C_2 consisting of d arcs. Then it is easy to find a one-parameter family M_s ($s \in [0, 1]$) of admissible multiarcs on C_2 that connects M to M' . Then $Gr_{M_s}(C_2) = C_1$ for all $s \in [0, 1]$. Therefore the choice of M is unique up to such an isotopy. \square

6.3. Fat traintracks.

Definition 6.6 (c.f. [15]). *Let F be a topological surface. A topological (fat) traintrack on F is a union $T = \cup_{j \in J} B_j$ of topological quadrangles $B_j \subset F$, such that, with some homeomorphisms $\phi_j: [-1, 1] \times [-1, 1] \rightarrow B_j \subset F$ ($j \in J$),*

- *there are locally finitely many quadrangles B_j on F ,*
- *the set of the outermost vertical edges $\phi_j(\{\pm 1\} \times [-1, 1])$ of B_j for all $j \in J$ are (disjointly) divided into triples $\{e_{k,1}, e_{k,2}, e_{k,3}\}_{k \in K}$ and pairs $\{e_{h,1}, e_{h,2}\}_{h \in H}$ so that*

- for each $k \in K$, there is a point p_k on $e_{k,1}$ that divides $e_{k,1}$ into $e_{k,2}$ and $e_{k,3}$, so that $e_{k,1} = e_{k,2} \cup e_{k,3}$,
- for each $h \in H$, we have $e_{h,1} = e_{h,2}$, and
- different quadrangles B_{j_1} and B_{j_2} may intersect only in their outermost vertical edges only in the way described above.

Each quadrangle B_j is called a branch of the traintrack T . The (vertical) arc $\phi_j(\{t\} \times [-1, 1])$ is called a tie of the branch B_j for each $t \in [-1, 1]$. The (horizontal) arc $\phi_j([0, 1] \times \{s\})$ is called a rail of the branch B_j for each $s \in [-1, 1]$. In particular, $\phi_j(\{\pm 1\} \times [-1, 1])$ are called outermost ties of B_j . For each $k \in K$, the point p_k is called a switch point of T . Then, each branch B_j have two distinct foliations: the foliation by the rails and by the ties of B_j . Then the entire traintrack $T = \cup B_j$ is foliated by the ties.

Definition 6.7. *Let λ be a lamination on the surface F . Then the topological traintrack $T = \cup_{j \in J} B_j$ carries λ , if the interior of T contains λ and each component of $B_j \cap \lambda$ is an arc connecting the outermost ties of B_j for each $j \in J$. If, in addition, $B_j \cap \lambda \neq \emptyset$ for all $j \in J$, then we say T fully carries λ .*

Suppose that $L = (\lambda, \mu)$ is a measured lamination carried by $T = \cup_{j \in J} B_j$. By $\mu(B_j) \in \mathbb{R}_{\geq 0}$, we mean the weight of L on the branch B_j , i.e. $\mu(B_j) := \mu(\alpha)$, where α is an arc in B_j intersecting each leaf of $L \cap B_j$ transversally in a single point.

Let r be a rail of a branch B_j and v be an endpoint of r . If v is a switch point, then it is identified with exactly two endpoints of other rails in adjacent branches. Otherwise, v is identified with exactly one endpoint of a different rail. Thus, identifying the corresponding endpoints of rails, we have a singular foliation of T where the singular points are exactly the switch points p_k .

Then a rail is an immersion r of \mathbb{R} into a leaf of this singular foliation of T , up to a homeomorphism of the domain, such that

- $r(t)$ does not converge as $t \rightarrow \infty, -\infty$ and
- if $r(t)$ is a switch point p_k for some $t \in \mathbb{R}$, letting $B_{k,1}, B_{k,2}, B_{k,3}$ denote the branches of T corresponding to the outermost ties $e_{k,1}, e_{k,2}, e_{k,3}$, respectively, for p_k as in Definition 6.6, then r embeds a small neighborhood of t in \mathbb{R} into either $B_{k,1} \cup B_{k,2}$ or $B_{k,1} \cup B_{k,3}$.

We assign a hyperbolic structure τ to the surface F . Then T is called a smooth (fat) traintrack if all $\phi_i: [-1, 1] \times [-1, 1] \rightarrow F$ are smooth and

all rails of T are smooth. The length of the branch B_j is the maximal length of the rails of B_j .

Definition 6.8 (c.f. [4]). *For $\epsilon > 0$, a smooth traintrack T on τ is called ϵ -straight if every tie and rail of T has curvature less than ϵ and, if a tie and a rail of T intersects at a point, the angle of intersection is ϵ -close to $\pi/2$. The traintrack T is called ϵ -slim if all ties have length less than ϵ .*

Definition 6.9. *We say that a geodesic lamination λ on τ is embedded in a smooth traintrack T , if λ is carried by T and each leaf of λ is a rail of T . Furthermore, λ is fully embedded in T , if, in addition, λ is fully carried by T .*

6.4. Decomposition of projective structures by traintracks. We state the main proposition for the proof of Theorem 6.1 (I).

Proposition 6.10. *Let $C = (\tau, L) = (f, \rho)$ be a projective structure on S . Then there exists a neighborhood U of $[L]$ in $\mathcal{PML}(S)$ such that, if a projective structure $C' = (\tau', L') = (f', \rho)$ on S with the same holonomy ρ satisfies $[L'] \in U \setminus \{[L]\}$, then there are a topological (fat) traintrack $T = \cup_{j=1}^n B_j$ on S and marking homeomorphisms $\phi: S \rightarrow C$, $\phi': S \rightarrow C'$ with the following properties:*

- (I) $\phi' \circ \phi^{-1}: C \rightarrow C'$ induces an isomorphism from $C \setminus \phi(T)$ to $C' \setminus \phi'(T)$ compatible with the developing maps f and f' .
- (II) (i) For all $j = 1, 2, \dots, n$, $C'|_{\phi'(B_j)}$ and $C|_{\phi(B_j)}$ are projective structures on a quadrangle supported on the same round cylinder with arcs and the outermost ties of B_j correspond the boundary components of the round cylinder,
- (ii) $C'|_{\phi'(B_j)} = Gr_{M_j}(C|_{\phi(B_j)})$ for some admissible multiarc M_j , and
- (iii) letting $\kappa: C \rightarrow \tau$ and $\kappa': C' \rightarrow \tau'$ be collapsing maps, the $\kappa \circ \phi$ -image of $T = \cup_{j=1}^n B_j$ is a topological traintrack carrying $L = (\lambda, \mu)$ and the $\kappa' \circ \phi'$ -image of $T = \cup_{j=1}^n B_j$ is a topological traintrack on τ' fully carrying $L' = (\lambda', \mu')$, and moreover $\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))$ is a good approximation of M_j .

We let $C = (\tau, L)$ be a projective structure on S with holonomy $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Let (C_i) be a sequence in $\mathcal{P}_\rho(S)$ such that, setting $C_i = (\tau_i, L_i)$, the projective measure lamination $[L_i]$ converges to $[L]$ in $\mathcal{PML}(S)$. Then, it suffices to show Proposition 6.10, for $C' = C_i$ with sufficiently large $i \in \mathbb{N}$. Set $L_i = (\lambda_i, \mu_i)$, where $\lambda_i \in \mathcal{GL}(\tau_i)$ and μ_i is a transversal measure supported on λ . Since $\mathcal{GL}(S)$

is compact, we can in addition assume that λ_i converges to a geodesic lamination λ_∞ (in $\mathcal{GL}(S)$). Then since $[L_i] \rightarrow [L]$ in $\mathcal{PML}(S)$, λ is a sublamination of λ_∞ .

6.5. Construction of traintracks on τ and τ' for Proposition 6.10 (I). Let $\kappa: C \rightarrow \tau$ and $\kappa_i: C_i \rightarrow \tau_i$ be the collapsing maps for all $i \in \mathbb{N}$.

Proposition 6.11. *There exists $K > 0$ with the following property: For every $\epsilon > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then there exist a fat traintrack $T = \cup_{j=1}^n B_j$ on S and marking homeomorphisms $\psi: S \rightarrow \tau$ and $\psi_i: S \rightarrow \tau_i$ such that*

- (i) *there is a marking-preserving smooth $(1 + \epsilon)$ -bilipschitz map $\eta_i: \tau \rightarrow \tau_i$ such that $\eta_i \circ \psi = \psi_i$ and η_i satisfies the conclusions of Proposition 5.4,*
- (ii) *$\psi(T) = \cup_{j=1}^n \psi(B_j)$ is an ϵ -straight and ϵ -slim smooth traintrack on τ such that the branches $\psi(B_j)$ have length more than K and the geodesic lamination λ_∞ is fully embedded into $\psi(T)$; similarly $\psi_i(T) = \cup_j \psi_i(B_j)$ is an ϵ -straight and ϵ -slim smooth traintrack on τ_i such that the branches $\psi_i(B_j)$ have length more than K and the geodesic lamination λ_i is fully embedded into $\psi_i(T)$ (here, for the embedding property of λ_∞ and λ_i , we may use different sets of homeomorphisms $\phi_j: [-1, 1] \times [-1, 1] \rightarrow B_j \subset S$), and*
- (iii) *$\kappa^{-1} \circ \psi(S \setminus T) \subset C$ is isomorphic to $\kappa_i^{-1} \circ \psi_i(S \setminus T) \subset C_i$ via developing maps.*

Moreover, if λ_∞ contains no closed leaf, we can take an arbitrary number $K > 0$ and otherwise we can take $K > 0$ to be one third of the length of the shortest closed leaf of λ_∞ .

Proof. Step1: Construction of T and $\psi: S \rightarrow \tau$.

Lemma 6.12. *Let ν be a geodesic lamination on a hyperbolic surface σ homeomorphic to S . There exists $K > 0$, such that, for every $\epsilon > 0$, there exists an ϵ -straight and ϵ -slim traintrack $T = \cup_j R_j$ on σ fully carrying ν such that*

- (i) *each branch of T has length more than K ,*
- (ii) *the ties of T are geodesic segments,*
- (iii) *each leaf of ν is a rail of T , and*

If ν contains a closed leaf, then we can take the constant K to be one third of the length of the shortest closed leaf of ν . Otherwise, we can take K to be any positive number.

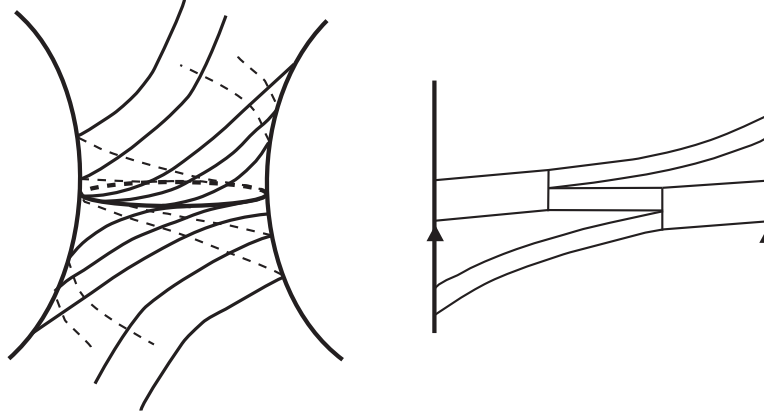


FIGURE 3.

Proof. Given $\epsilon > 0$, if $\delta > 0$ is sufficiently small, we can easily make $N_\delta(\nu)$ into an ϵ -straight and ϵ -slim traintrack satisfying (ii) and (iii); let $T_\delta = N_\delta(\nu) = \cup_{j=1}^n B_j$ denote this traintrack (with some $n = n(\delta) \in \mathbb{N}$).

If ν contains a close leaf l , then any traintrack T carrying ν must have a branch of length equal to or less than $\text{length}_\sigma(l)$. If, in addition, if some leaves of ν spiral towards l from the both sides of l , then l must be covered by at least 2 branches of T (Figure 3). Thus T must have a branch of length equal to or less than $\frac{1}{2}\text{length}_\sigma(l)$. The number of the switch points of T_δ is bounded for all $\delta > 0$, and thus we can assume that the number $n = n(\delta)$ of the branches of T_δ is also bounded. On the other hand, if ν contains a non-periodic leaf, the total length of the branches B_j of T_δ diverges to infinity, as $\delta \rightarrow 0$. Therefore, for every $K > 0$, we can split the traintrack T_δ appropriately so that the length of every branch disjoint from periodic leaves of ν is more than K , keeping (ii) and (iii).

Hence we can take K to be one third of the length of the shortest closed leaf of ν . \square

Let $T_\infty = \cup_{j=1}^n R_j$ denote the ϵ -slim traintrack on τ carrying λ_∞ , obtained by Lemma 6.12, where R_j are branches of T_∞ . Then pick a topological traintrack $T = \cup B_j$ on S so that (S, T) is topologically isomorphic to (τ, T_∞) via a marking homeomorphism $\psi: S \rightarrow \tau$ of τ and that $\psi(B_j) = R_j$ for each $j = 1, 2, \dots, n$.

Let \mathcal{L} be the canonical measured lamination on C corresponding to L via $\kappa: C \rightarrow \tau$ (§3.2.2). Then, since $\tau \setminus T_\infty$ is disjoint from λ_∞ , κ^{-1} embeds $\tau \setminus T_\infty$ onto its image. Let $\mathcal{T}_\infty = \kappa^{-1}(T_\infty)$, which is, at the moment, just a subset of C . Fix a marking $\phi: S \rightarrow C$ of C that takes

T to T_∞ . Then, since ψ is a homeomorphism, there exists a unique continuous map $\zeta: S \rightarrow S$ such that ζ preserves T (as a subset of S) and $\psi \circ \zeta = \kappa \circ \phi$. Note that, since ϕ and ψ are homeomorphisms, ζ is topologically equivalent to κ .

Step 2: Proposition 6.11 (i) and (iii). Since λ_∞ is contained in $\text{int}(T_\infty)$, we can pick $\delta > 0$ so that $N_{2\delta}(\lambda_\infty) \subset T_\infty$. Let $\eta_i: \tau \rightarrow \tau_i$ be the $(1 + \epsilon)$ -bilipschitz map obtained by Proposition 5.4. Then the η_i^{-1} -image of $\tau_i \setminus N_\delta(\lambda_i)$ converges to $\tau_\infty \setminus N_\delta(\lambda_\infty)$ as $i \rightarrow \infty$. Since $(\tau_\infty \setminus N_\delta(\lambda_\infty)) \cap \lambda_\infty = \emptyset$, Proposition 5.4, $\beta_i \circ \tilde{\eta}_i$ converges to β , in C^1 -topology, on the total lift of $\tau \setminus N_\delta(\lambda_\infty)$ to \mathbb{H}^2 , where $\tilde{\eta}_i: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is the lift of $\eta_i: \tau \rightarrow \tau_i$. In addition, since C and C_i have the same orientation, the normal vector of $\beta_i \circ \tilde{\eta}_i$ converges to that of β on the total lift. Thus $\kappa_i^{-1}(\tau_i \setminus N_\delta(\lambda_i)) \subset C_i$ converges to $\kappa^{-1}(\tau \setminus N_\delta(\lambda_\infty)) \subset C$ via the developing maps f and f_i as $i \rightarrow \infty$. Therefore, since $\kappa^{-1}(\tau \setminus \lambda_\infty)$ is a regular neighborhood of $\kappa^{-1}(\tau \setminus N_\delta(\lambda_\infty))$, for sufficiently large i , there is a canonical isomorphic embedding $\iota_i: \kappa_i^{-1}(\tau_i \setminus N_\delta(\lambda_i)) \rightarrow \kappa^{-1}(\tau \setminus \lambda_\infty)$ compatible with f_i and f . Thus, since $\tau \setminus N_\delta(\lambda_\infty)$ is a regular neighborhood of $\tau \setminus N_{2\delta}(\lambda_\infty)$, we can assume that $\text{Im}(\iota_i)$ contains $\kappa^{-1}(\tau \setminus N_{2\delta}(\lambda_\infty))$. Thus we can pick a marking homeomorphism $\phi_i: S \rightarrow C_i$ of C_i so that $\phi_i(S \setminus T)$ is contained in the domain of ι_i and $\iota_i \circ \phi_i = \phi$ on $S \setminus T$. In addition, by perturbing $\eta_i: \tau \rightarrow \tau_i$ (but preserving the properties from Proposition 5.4), we can assume that $\eta_i|_{\psi(S \setminus T)}$ corresponds to the isomorphism $\iota_i^{-1}: C \setminus \phi(T) \rightarrow C_i \setminus \phi_i(T)$ via κ and κ_i , i.e. $\kappa_i \circ \iota_i^{-1} = \eta_i \circ \kappa$ on $\phi(S \setminus T)$. Set $\psi_i = \eta_i \circ \psi: S \rightarrow \tau_i$ for each $i \in \mathbb{N}$. Thus we have constructed ψ and ψ_i satisfying (iii) and η_i satisfying (i).

Step 3: (ii). We have already constructed a desired $\psi: S \rightarrow \tau$ in Step 1. In Step 2, we set $\psi_i = \eta_i \circ \psi: S \rightarrow \tau_i$. Since $\eta_i: \tau \rightarrow \tau_i$ converges to the trivial isometry and (τ_i, λ_i) converges to (τ, λ_∞) as $i \rightarrow \infty$, the properties of ψ induce the desired properties for ψ_i . \square

Pick sufficiently small $\epsilon > 0$. Let $T_\infty = \cup_j R_j$ denote the ϵ -straight and slim traintrack on τ obtained by Proposition 6.11, where R_j ($j = 1, 2, \dots, n$) are branches of T_∞ . Then T_∞ fully carries λ_∞ and each branch R_j has length more than K , where $K > 0$ is a constant which does *not* depend on ϵ . Let $\tilde{\lambda}_\infty$ and \tilde{T}_∞ denote the total lifts of λ_∞ and T to \mathbb{H}^2 , respectively. Then \tilde{T}_∞ is an ϵ -straight and slim traintrack carrying $\tilde{\lambda}_\infty$.

Lemma 6.13. *For every $\delta > 0$, there exists $e > 0$ such that, if $\epsilon < e$, then, for all leaves l, m of $\tilde{\lambda}_\infty \in \mathcal{GL}(\mathbb{H}^2)$ passing through the same branch R of \tilde{T}_∞ , $\beta(l \cap R)$ and $\beta(m \cap R)$ are geodesics segments of length more than K in \mathbb{H}^3 that are δ -close in Hausdorff topology.*

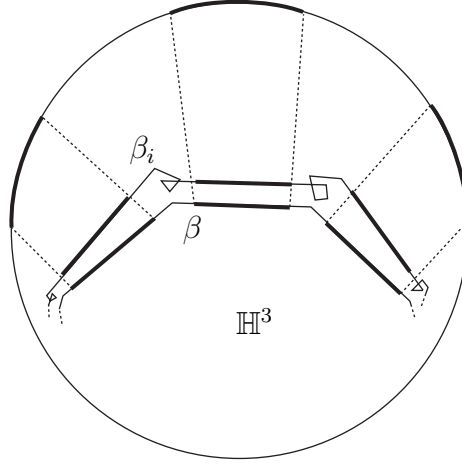


FIGURE 4. C^1 -close subsurfaces of the bending maps β and β_i correspond to isomorphic subsurfaces in C and C_i .

Proof. Fix $\delta > 0$. Since all branches of T_∞ have length more than K , for all leaves l, m of $\tilde{\lambda}_\infty$ passing through a fixed branch R of \tilde{T}_∞ , $l \cap R$ and $m \cap R$ are geodesic segments in \mathbb{H}^2 of length more than K . Since \tilde{T}_∞ is ϵ -slim, corresponding endpoints of $l \cap R$ and $m \cap R$ have distance less than ϵ . Thus, by taking sufficiently small $\epsilon > 0$, we can assume that $l \cap R$ and $m \cap R$ are δ -close in Hausdorff topology for all R, l, m as above. Recall that β is 1-lipschitz and β isometrically embeds leaves of $\tilde{\lambda}_\infty$ onto geodesics in \mathbb{H}^3 . Therefore, since $l \cap R$ and $m \cap R$ are δ -close, $\beta(l \cap R)$ and $\beta(m \cap R)$ are also δ -close in Hausdorff topology and they are geodesic segments of length more than K . \square

For sufficiently large $i \in \mathbb{N}$, let $T_i = \cup_{j=1}^n R_{i,j}$ is the ϵ -straight and slim traintrack on τ_i obtained by Proposition 6.11. Then each branch R_j has length more than K . Similarly let $\tilde{\lambda}_i$ and \tilde{T}_i denote the total lifts of λ_i and T_i , respectively, to \mathbb{H}^2 . Then \tilde{T}_i is an ϵ -straight and slim, $\tilde{\lambda}_i$ fully embeds into \tilde{T}_i and each branch of \tilde{T}_i has length at least K . Then, similarly

Lemma 6.14. *For every $\delta > 0$, there exists $e > 0$ with the following property. If $\epsilon < e$, then for arbitrary leaves l, m of $\tilde{\lambda}_i$ that intersects the same branch R of \tilde{T}_i , which is ϵ -straight and slim, $\beta(l \cap R)$ and $\beta(m \cap \tilde{R}_i)$ are geodesics segments of length more than K (in \mathbb{H}^3) that are δ -close in Hausdorff topology.*

Proof. The proof is similar to the proof of Lemma 6.13. \square

Recall that traintracks $T_\infty = \cup_{j=1}^n R_j$ on τ and $T_i = \cup_{j=1}^n R_{i,j}$ on τ_i are the images of $T = \cup_{j=1}^n B_j$ on S under $\psi: S \rightarrow \tau$ and $\psi_i: S \rightarrow \tau_i$ (Proposition 6.11 (ii)). Let \tilde{T} be the total lift of T to \mathbb{H}^2 , and let $\tilde{\psi}: \tilde{S} \rightarrow \mathbb{H}^2$ and $\tilde{\psi}_i: \tilde{S} \rightarrow \mathbb{H}^2$ be the lifts of ψ and ψ_i , respectively. Then we have

Lemma 6.15. *For every $\delta > 0$, there exist $e > 0$ and $I \in \mathbb{N}$ such that, if T_∞ and T_i are ϵ -straight and slim with $\epsilon < e$ and $i > I$, then, for every branch B of \tilde{T} and every leaf l of $\tilde{\lambda}_\infty$ intersecting the branch $\tilde{\psi}(B)$ of \tilde{T}_∞ and leaf m of $\tilde{\lambda}_i$ intersecting the branch $\tilde{\psi}_i(B)$ of \tilde{T}_i , the geodesic segments $\beta(l \cap \tilde{\psi}(B))$ and $\beta_i(m \cap \tilde{\psi}_i(B))$ are of length more than K and they are δ -close with the Hausdorff metric.*

Proof. Assuming $i \in \mathbb{N}$ is sufficiently large, since every branch of T_∞ and T_i has length at least K , then $\beta(l \cap \tilde{\psi}(B))$ and $\beta_i(m \cap \tilde{\psi}_i(B))$ are geodesic segments in \mathbb{H}^3 of length more than K (for all l, m, B as in the statements).

By Proposition 6.11 (and Proposition 5.4), for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\beta_i \circ \tilde{\eta}_i$ and β are δ -close in C^0 -topology. In addition, since \tilde{T}_∞ and \tilde{T}_i are ϵ -slim, for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then corresponding end points of $m \cap \tilde{\psi}_i(B)$ and $\tilde{\eta}_i(l \cap \tilde{\psi}(B))$ are δ -close. Thus, under the same assumption, corresponding endpoints of the geodesic segments $\beta(m \cap \tilde{\psi}_i(B))$ and $\beta_i(l \cap \tilde{\psi}(B))$ are δ -close. \square

Lemma 6.16. *For every $N > 0$, there exists $I \in \mathbb{N}$ such that, if $[L] \neq [L_i]$ and $i > I$, then $\mu_i \circ \psi_i(B) > N \cdot \mu \circ \psi(B)$ for every branch B of T .*

Proof. We have set $T = \cup_{j=1}^n B_j$, where n is the number of branches of T . Recall that T corresponds to T_i and T_∞ via ψ_i and ψ , respectively. Thus, since L_i is fully carried by T_i , L_i is identified with the n -tuple $(\mu_i \circ \psi_i(B_j))_{j=1}^n \in (\mathbb{R}_{>0})^n$. Similarly, since L is carried by T_∞ , and it is identified with the n -tuple $(\mu \circ \psi(B_j))_{j=1}^n \in (\mathbb{R}_{\geq 0})^n$. Since $[L_i]$ converges to $[L]$ in $\mathcal{PML}(S)$ as $i \rightarrow \infty$, then, for every $\delta > 0$, if i is sufficiently large, we have

$$(2) \quad (1 - \delta) \frac{\mu \circ \psi(B_j)}{\mu \circ \psi(B_k)} \leq \frac{\mu_i \circ \psi_i(B_j)}{\mu_i \circ \psi_i(B_k)} \leq (1 + \delta) \frac{\mu \circ \psi(B_j)}{\mu \circ \psi(B_k)}$$

for all branches B_j and B_k of T .

Suppose that the lemma fails. Then, taking a subsequence of $(C_i)_i$ if necessarily, we can assume that $\mu_i \circ \psi_i(B_j)$ indexed by $i \in \mathbb{N}$ is a bounded sequence in $\mathbb{R}_{>0}$ for some $j \in \{1, 2, \dots, n\}$. Thus, up to a subsequence, by Inequalities (2), the sequence $\mu_i \circ \psi_i(B_j)$ converges as

$i \rightarrow \infty$ for every $j = 1, 2, \dots, n$. Then, since $\delta > 0$ is arbitrarily fixed, by Inequalities (2), L_i converges to some $c \cdot L$ with $c > 0$ as $i \rightarrow \infty$. In addition, we can assume that all L_i are distinct. Since τ_i converges to τ , thus $C_i = (\tau_i, L_i) \in \mathcal{P}_\rho$ converges to (τ, cL) in $\mathcal{P}(S)$. This is a contradiction to the fact that \mathcal{P}_ρ is a discrete subset of $\mathcal{P}(S)$. 6.16

6.6. Branches supported on a cylinder. Given any $\epsilon > 0$, we have constructed an ϵ -straight and slim traintrack $T_\infty (= T_\infty(\epsilon)) = \cup_{j=1}^n R_j$ on τ into which λ_∞ is embedded (Proposition 6.11). In addition, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, we can assume the conclusions in Lemma 6.13, Lemma 6.14, and Lemma 6.15. Since λ_∞ is embedded in T_∞ , all ties of T_∞ are transversal to λ_∞ and thus, in particular, to λ . Therefore, recalling the collapsing map $\kappa: C \rightarrow \tau$, for all ties t of T_∞ , $\kappa^{-1}(t)$ is a simple arc on C . Thus we can pull T_∞ back to a topological (fat) traintrack on C via κ . Namely, let $\mathcal{T}_\infty = \kappa^{-1}(T_\infty)$ and $\mathcal{R}_j = \kappa^{-1}(R_j)$ for each $j = 1, 2, \dots, n$. Then $\mathcal{T}_\infty = \cup_{j=1}^n \mathcal{R}_j$ is a topological traintrack on C that homeomorphic to $T_\infty = \cup_{j=1}^n R_j$ on τ . Note that we have also pulled back the rails and the ties of T_∞ to those of \mathcal{T} so that \mathcal{L} embeds into \mathcal{T} via κ , where \mathcal{L} is the canonical measured lamination on C .

Proposition 6.17. *We can perturb the traintrack structure $\mathcal{T}_\infty = \cup_j \mathcal{R}_j$ (more precisely, branches and ties) so that the projective structure on each branch is supported on a round cylinder but \mathcal{T}_∞ is preserved a subset of C .*

We prove this proposition in the rest of §6.6.

6.6.1. Round circles on $\hat{\mathbb{C}}$ corresponding to switches of T_∞ . If two adjacent branches of the traintrack T have a common outermost tie, then there is a no switch point on this outermost tie (this situation can be avoided if λ_∞ contains no isolated periodic leaf). For each such an outermost tie t of (a branch of) T , pick an endpoint of t and call it marked point of the traintrack T . Accordingly, we call the corresponding point on T_i and T_∞ a marked point as well. Thus, every outermost tie of T, T_∞ and T_i contains a unique switch point or marked point.

Recall that $\tilde{T}_\infty = \tilde{T}_\infty(\epsilon)$ is the total lift of the traintrack T_∞ on τ to \mathbb{H}^2 . Letting $\tilde{\kappa}: \tilde{C} \rightarrow \mathbb{H}^2$ be the lift of $\kappa: C \rightarrow \tau$, we have

Lemma 6.18. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then then, for each switch point and marked point p of $\tilde{T}_\infty(\epsilon)$, there exists a round circle c_p on $\hat{\mathbb{C}}$ satisfying the followings:*

- (i) *This assignment of round circles c_p to switch points and marked points p of $\partial\tilde{T}_\infty$ is ρ -equivariant.*
- (ii) *c_p contains the point $f(\tilde{\kappa}^{-1}(p))$.*
- (iii) *If a leaf l of $\tilde{\lambda}_\infty$ passes through the tie of \tilde{T}_∞ containing p , then the geodesic $\beta(l)$ intersects the totally geodesic hyperplane $\text{Conv}(c_p)$ transversely at an angle δ -close to $\pi/2$.*
- (iv) *Letting q be the point on l that maps into $\text{Conv}(c_p)$, then we have $\text{dist}_{\mathbb{H}^2}(p, q) < \delta$.*

Proof. Let p be a switch point of \tilde{T}_∞ . Then, since $\text{int}(\tilde{T}_\infty)$ contains $\tilde{\lambda}_\infty$, then $p \cap \lambda_\infty = \emptyset$. Thus $\tilde{\kappa}^{-1}(p)$ is a single point on $\hat{\mathbb{C}}$. Then $f(\tilde{\kappa}^{-1}(p)) \in \hat{\mathbb{C}}$ orthogonally projects $\beta(p)$ in the hyperbolic tangent plane of β at p . Let a_p be the tie of \tilde{T}_∞ whose interior contains p , which is a union of two outermost ties of different branches of \tilde{T}_∞ . Since a_p is a geodesic segment on \mathbb{H}^2 (Lemma 6.12), β embeds a small neighborhood of p in a_p onto a geodesic segment in \mathbb{H}^3 . Thus we let c_p be the round circle on $\hat{\mathbb{C}}$ containing the point $f(\tilde{\kappa}^{-1}(p))$ such that the hyperplane $\text{Conv}_{\mathbb{H}^3}(c_p)$ contains this geodesic segment (Condition (ii)). The $\pi_1(S)$ -action on the universal cover of τ preserves the set of switch points on \tilde{T}_∞ . Then, since β is ρ -equivariant, this assignment of c_p for switch points p of \tilde{T}_∞ is accordingly ρ -equivariant (Condition (i)).

Let m_p be the leaf of $\tilde{\lambda}_\infty$ closest to p . Since \tilde{T}_∞ is ϵ -straight, m_p intersects a_p transversely, and $\angle(m_p, a_p)$ is ϵ -close to $\pi/2$. Thus, if $0 < \epsilon < \delta/2$, the geodesic $\beta(m_p)$ intersects $\text{Conv}_{\mathbb{H}^3}(c_p)$ at an angle $\delta/2$ -close to $\pi/2$ for all switch points p of \tilde{T}_∞ . Thus, by lemma 6.13, if $\epsilon > 0$ is sufficiently small, then, for each leaf l of $\tilde{\lambda}_\infty$ intersecting a_p , $\beta(l)$ transversally intersects $\text{Conv}(c_p)$ at an angle δ -close to $\pi/2$ (Condition (iii)). Since $\text{length}(a_p) < \epsilon$, the distance between the points p and $l \cap a_p$ is also less than ϵ . Then, since β is 1-lipschitz, the distance between $\beta(p)$ and $\beta(l \cap a_p)$ is also less than ϵ . Let q_l be the point on l such that $\beta(q_l) \in \text{Conv}(c_p)$. Then, since $\angle(\beta(l), \text{Conv}(c_p))$ is δ -close to $\pi/2$ for all l and p as above, if $\epsilon > 0$ is sufficiently small, then, in addition $\text{dist}_{\mathbb{H}^3}(\beta(q_l), \beta(l \cap a_p)) = \text{dist}_{\mathbb{H}^2}(q_l, l \cap a_p)$ is less than $\delta/2$ for all l as above. Therefore, by the triangle inequality, $\text{dist}_{\mathbb{H}^2}(p, q_l) < \delta/2 + \epsilon < \delta$ (Condition (vi)).

Let p be a marked point on \tilde{T}_∞ ; then we can similarly define c_p and show (i) - (iv). Then $\tilde{\kappa}^{-1}(p)$ is a single point on $\hat{\mathbb{C}}$. Let a_p be the tie of \tilde{T}_∞ whose endpoint is p , which is a geodesic segment in \mathbb{H}^2 . Since the interior of \tilde{T}_∞ contains \tilde{L} , then β takes a sufficiently short subsegment of a_p with an endpoint at p onto a geodesic segment in \mathbb{H}^3 . Then, let

c_p be the round circle on $\hat{\mathbb{C}}$ containing $f(\tilde{\kappa}^{-1}(p))$ such that $\text{Conv}(c_p)$ contains the image of this subsegment. Then c_p satisfies (ii), and we similarly see that it satisfies the other conditions as well. \square

Let l be a leaf of $\tilde{\lambda}_\infty$, and let a_1 and a_2 be different outermost ties of branches of \tilde{T}_∞ that intersect l . Accordingly, let p_1 and p_2 be the (different) switch or marked points on $\partial\tilde{T}_\infty$ contained in a_1 and a_2 , respectively; let c_1 and c_2 be the round circles on $\hat{\mathbb{C}}$ corresponding to p_1 and p_2 given by Lemma 6.18; let q_1 and q_2 be the distinct points on l such that l intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ at $\beta(q_1)$ and $\beta(q_2)$. In addition, let $P_1, P_2 \in \hat{\mathbb{C}}$ denote the ideal endpoints of the geodesic $\beta(l) \subset \mathbb{H}^3$ so that the geodesic rays from $\beta(q_1)$ to P_1 and from $\beta(q_2)$ to P_2 are disjoint. Recalling that c_1, c_2 depend on the choice of $\delta > 0$ in Lemma 6.18, we have

Lemma 6.19. *There exists (sufficiently small) $\delta > 0$ and $\epsilon > 0$ such that, for all leaves l of $\tilde{\lambda}_\infty$ and all outermost ties a_1 and a_2 of branches of $\tilde{T}_\infty = \tilde{T}_\infty(\epsilon)$ such that a_1 and a_2 intersect l ,*

- *the corresponding round circles c_1 and c_2 on $\hat{\mathbb{C}}$ are disjoint, and*
- *P_k is in the disk component of $\hat{\mathbb{C}} \setminus (c_1 \cup c_2)$ bounded by c_k for each $k = 1, 2$.*

Proof. Recall that there is a lower bound $K > 0$ of the lengths of branches of $T_\infty(\epsilon)$, which does *not* depend on $\epsilon > 0$. Then the distance between the points $a_1 \cap l$ and $a_2 \cap l$ is more than K . By Lemma 6.18, we have $\text{dist}(q_k, p_k) < \delta$ for each $k = 1, 2$. Thus $\text{dist}(q_1, q_2) > K - 2\delta$. In addition, again by Lemma 6.18, $\beta(l)$ intersects $\text{Conv}(c_{p_k})$ at an angle δ -close to $\pi/2$. Therefore, for sufficiently small $\delta > 0$, c_{p_1} and c_{p_2} are disjoint round circles on $\hat{\mathbb{C}}$ for all l, a_1 and a_2 as in the lemma. Observing the configuration of $\beta(l)$, c_{p_1} and c_{p_2} in \mathbb{H}^3 , clearly the ideal point P_k is in the disk component of $\hat{\mathbb{C}} \setminus (c_1 \cup c_2)$ bounded by c_k for each $k = 1, 2$. \square

More generally, let l be a rail of \tilde{T}_∞ . Then consider all outermost ties a_k ($k \in \mathbb{Z}$) intersecting l . We can naturally assume that the intersection points $a_k \cap l$ lie in l in the order of the index $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, let c_k be the round circle, given by Lemma 6.18, corresponding to the switch point or the marked point of \tilde{T}_∞ contained in a_k . Since \tilde{T}_∞ fully carries $\tilde{\lambda}_\infty$, for every k , there is a leaf of $\tilde{\lambda}_\infty$ that intersects a_{k-1}, a_k, a_{k+1} . Then, by Lemma 6.19, c_{k-1}, c_k, c_{k+1} are disjoint round circles on $\hat{\mathbb{C}}$ nested in the listed order. Therefore

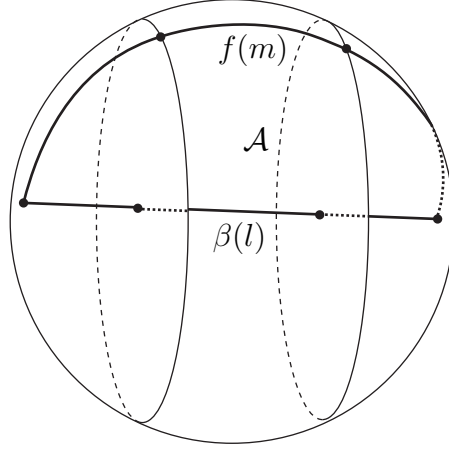


FIGURE 5.

Corollary 6.20. *The round circles c_k are disjoint and nested in the order of the index $k \in \mathbb{Z}$. That is, $\hat{\mathbb{C}} \setminus (\cup_k c_k \cup \{P_1, P_2\})$ is the union of disjoint round cylinders bounded by c_k and c_{k+1} for all $k \in \mathbb{Z}$.*

Recalling that λ is a sublamination of λ_∞ , let ν be the geodesic lamination on C with respect to Thurston's metric that is the union of the supporting lamination of the canonical lamination \mathcal{L} and $\kappa^{-1}(\lambda_\infty \setminus \lambda)$. Then ν fully embeds into \mathcal{T}_∞ .

Let R and \mathcal{R} be corresponding branches of \tilde{T}_∞ and $\tilde{\mathcal{T}}_\infty$, respectively. Recall that every outermost tie of a branch contains a switch point or marked point. Let c_1 and c_2 denote the round circles on $\hat{\mathbb{C}}$, given by Lemma 6.18, corresponding to the switches on the outermost ties of R . Let \mathcal{A} denote the round annulus bounded by c_1 and c_2 . Then we have

Proposition 6.21. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then for every rail m of $\tilde{\mathcal{T}}_\infty$ intersecting the branch \mathcal{R} of $\tilde{\mathcal{T}}_\infty$, letting l be the corresponding rail of \tilde{T}_∞ , we have*

- $f|m$ is a simple curve on $\hat{\mathbb{C}}$ that connects the ideal ends points of $\beta(l)$ and intersects both c_1 and c_2 in a single point at an angle δ -close to $\pi/2$; thus $f^{-1}(\mathcal{A}) \cap m$ is a single arc, and
- the arc $f^{-1}(\mathcal{A}) \cap m$ is δ -close to the arc $\mathcal{R} \cap m$ with the Hausdorff metric.

Proof. Let $\tilde{\nu} \in \mathcal{GL}(\tilde{C})$ be the total lift of the geodesic lamination ν on C to \tilde{C} . (Case 1.) We first prove the proposition for leaves of $\tilde{\nu}$. For each leaf m of $\tilde{\nu}$, $f|m$ is a circular arc on $\hat{\mathbb{C}}$ connecting the endpoints

of the geodesic $\beta(l)$ in \mathbb{H}^3 , where l is the leaf of $\tilde{\lambda}_\infty$ corresponding to m . Then the nearest point projection from $f(m)$ to $\beta(l)$ corresponds to $\tilde{\kappa}|m: m \rightarrow l$.

By Lemma 6.18 (iii) and (vi), for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, for every leaf l of λ_∞ passing through R , the geodesic $\beta(l)$ intersects the hyperplanes $\text{Conv}_{\mathbb{H}^3}(c_1)$ and $\text{Conv}_{\mathbb{H}^3}(c_2)$ at an angle δ -close to $\pi/2$ and, in addition, $R \cap l$ is δ -close to the geodesic segment $\beta^{-1}(\text{Conv}(\mathcal{A})) \cap l$.

Those properties for l induce the desired properties for m .

(Case 2.) Suppose m is a rail of $\tilde{\mathcal{T}}_\infty$ intersecting \mathcal{R} that is *not* a leaf of $\tilde{\nu}$. Then m is contained in a component \mathcal{X} of $\tilde{C} \setminus \tilde{\nu}$. Let m' be a leaf of $\tilde{\nu}$ bounding \mathcal{X} and passing through \mathcal{R} . We have seen that m' has the desired property. Roughly, since T_∞ is ϵ -straight and slim, then l and l' are sufficiently close and thus m also has the desired property, where l' is the leaf of $\tilde{\lambda}_\infty$ corresponding to m' .

Letting $X = \tilde{\kappa}(\mathcal{X})$, a component of $\mathbb{H}^2 \setminus \tilde{\lambda}_\infty$.

Let $\Psi_X: D_X \rightarrow H_X$ be the orthogonal projection corresponding to X , where D_X is the f -image of the maximal in \tilde{C} containing \mathcal{X} and H_X is the totally geodesic hyperplane in \mathbb{H}^3 containing $\beta(X)$. Then we have

$$(3) \quad \Psi \circ f = \beta \circ \tilde{\kappa}$$

on \mathcal{X} (see §3.2).

Since T_∞ is ϵ -straight, then l has curvature less than ϵ . For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then the rails $l \cap R$ and $l' \cap R$ of R are δ -close for all l, l' as above. Since β isometrically embeds X into D_X , then, if $\epsilon > 0$ is sufficiently small, then $\beta(l)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ in a single point δ -orthogonally for all rails l of \tilde{T}_∞ intersecting R . Thus we can in addition assume that $R \cap l$ is δ -close to $\beta^{-1}(\text{Conv}(\mathcal{A})) \cap l$. Those properties of l induce the desired properties of m , by (3). \square

For each rail m of the traintrack $\tilde{\mathcal{T}}_\infty$ passing through \mathcal{R} , let $\alpha_m = m \cap \mathcal{R}$, a rail of \mathcal{R} . Let $\alpha'_m = m \cap f^{-1}(\mathcal{A})$. Then, by Proposition 6.21, α'_m is a simple arc in m that is δ -close to $m \cap \mathcal{R}$, and α'_m is supported on \mathcal{A} . Suppose that m contains a switch point \mathbf{p}_k on $\partial\mathcal{R}$ that corresponds to a switch point p_k of ∂R via $\tilde{\kappa}$. Then \mathbf{p}_k is an endpoint of α_m . Since the round circle c_k corresponding the switch point p_k passes through \mathbf{p}_k by Lemma 6.18 (ii), the δ -perturbation of α_m to α'_m preserves the endpoint \mathbf{p}_k . In addition, since the developing map f is a local homeomorphism, α'_m changes continuously when we vary the leaf m passing through \mathcal{R} continuously. Set $\mathcal{R} = \cup \alpha_m$ and

$\mathcal{R}' = \cup \alpha'_m$, where the union runs over all rails m of \mathcal{T}_∞ passing through \mathcal{R} . Therefore

Corollary 6.22. *\mathcal{R}' is a quadrangle supported on \mathcal{A} , that is δ -close to \mathcal{R} with the Hausdorff metric, and this deformation preserves the switch point of $\tilde{\mathcal{T}}_\infty$ on $\partial\mathcal{R}$.*

We last show that this deformation of \mathcal{R} to \mathcal{R}' induces a desired deformation of the traintrack T_∞ . By Lemma 6.18 (i), the correspondence between the branches \mathcal{R} of $\tilde{\mathcal{T}}_\infty$ and the round cylinders \mathcal{A} in $\hat{\mathbb{C}}$ is ρ -equivariant. Thus this deformation of \mathcal{R} to \mathcal{R}' commutes with the action of $\pi_1(S)$ on \tilde{C} . Pick an arbitrary rail m of $\tilde{\mathcal{T}}_\infty$. Let $\{\mathcal{R}_k\}_{k \in \mathbb{Z}}$ denote the branches of $\tilde{\mathcal{T}}_\infty = \cup_j \mathcal{R}_j$ intersecting m . We can assume that this sequence is indexed along m so that \mathcal{R}_k and \mathcal{R}_{k+1} are adjacent for every $k \in \mathbb{Z}$. Then we have a decomposition of m into rails $m \cap \mathcal{R}_k$ of \mathcal{R}_k . For each $k \in \mathbb{Z}$, let \mathcal{A}_k be the round cylinder on $\hat{\mathbb{C}}$ corresponding to \mathcal{R}_k as for Proposition 6.21. Thus, by Proposition 6.21, $m \cap f^{-1}(\mathcal{A}_k)$ is a segment of m that is δ -close to $m \cap \mathcal{R}_k$ for each $k \in \mathbb{Z}$. Then, since \mathcal{A}_k have disjoint interior, m is decomposed into the rails $m \cap f^{-1}(\mathcal{A}_k)$ of \mathcal{R}'_k ($k \in \mathbb{Z}$). Then, by Corollary 6.22, the decomposition of m into the rails $m \cap \mathcal{R}_k$ of \mathcal{R} is δ -close to that into the rails $m \cap f^{-1}(\mathcal{A}_k)$ of \mathcal{R}'_k . Since this decomposition holds for all rails m of $\tilde{\mathcal{T}}_\infty$ and the deformation preserves m , if \mathcal{R} and \mathcal{Q} are different branches of $\tilde{\mathcal{T}}_\infty$, their deformations \mathcal{R}' and \mathcal{Q}' given by Corollary 6.22 have disjoint interiors. Thus we have a deformation of $\tilde{\mathcal{T}}_\infty = \cup \mathcal{R}_j$, preserving $\tilde{\mathcal{T}}_\infty$ as a subset of \tilde{C} , such that each branch is a quadrangle supported on a round cylinder in $\hat{\mathbb{C}}$. Since the deformation of each branch commutes with the action of $\pi_1(S)$, this deformation of $\tilde{\mathcal{T}}_\infty$ descends to a desired deformation of \mathcal{T}_∞ . We have thus completed the proof of Proposition 6.17.

6.7. Traintrack on C_i for Proposition 6.11 (ii). Similarly, we pull back the traintrack T_i on τ_i to a traintrack on C_i via $\kappa_i: C_i \rightarrow \tau_i$. Let $\mathcal{T}_i = \kappa_i^{-1}(T_i)$ and, for each $j = 1, 2, \dots, n$, let $\mathcal{R}_{i,j} = \kappa_i^{-1}(R_{i,j})$. Then, similarly to the case of \mathcal{T}_∞ , we see that $\mathcal{T}_i = \cup_{j=1}^n \mathcal{R}_{i,j}$ is a topological traintrack on C_i and the canonical geodesic lamination ν_i on C_i , which corresponds to λ_i via $\kappa_i: C_i \rightarrow \tau_i$, fully embeds into \mathcal{T}_i . Let $\tilde{\mathcal{T}}_i$ be the total lift of \mathcal{T}_i to \tilde{C}_i . In Proposition 6.17, we have given an appropriate deformation of \mathcal{T}_∞ that yields the traintrack $\phi(T)$ on C in Proposition 6.10 (II). We next construct an analogous deformation of \mathcal{T}_i to complete the proof of Proposition 6.10 (II):

Proposition 6.23. *For sufficiently large $i \in \mathbb{N}$, there is a small deformation of topological traintrack $\mathcal{T}_i = \cup_{j=1}^n \mathcal{R}_{i,j}$ such that*

- (i) this perturbation preserves \mathcal{T}_i as a subset of C_i , and
- (ii) letting \mathcal{R}_i and \mathcal{R} be the corresponding branches of $\tilde{\mathcal{T}}_i$ and $\tilde{\mathcal{T}}_\infty$, respectively, then \mathcal{R}_i is a rectangle supported on a round cylinder on $\hat{\mathbb{C}}$ whose support is the support of \mathcal{R} .

Proof. (Recall that the proof of Proposition 6.17 is done by observing the bending map $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$. We have proved that $\beta_i: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ converges to β as $i \rightarrow \infty$ (Theorem 5.4). Thus we will prove Proposition 6.23 by imitating the proof of Proposition 6.17 for sufficiently large i .)

Let \tilde{C}_i be the universal cover of C_i . Then let \mathcal{R}_i be a branch of $\tilde{\mathcal{T}}_i$, and let R_i be the branch of \tilde{T}_i corresponding to \mathcal{R}_i via $\tilde{\kappa}_i: \tilde{C}_i \rightarrow \mathbb{H}^2$. Accordingly let \mathcal{R} be the branch of $\tilde{\mathcal{T}}_\infty$ corresponding to \mathcal{R}_i , and let R be the branch of \tilde{T} corresponding to \mathcal{R} . Thus let \mathcal{A} be the round cylinder supporting the quadrangle $\mathcal{R} \subset \tilde{C}$. Then, recall that \mathcal{A} is bounded by the round circles c_1 and c_2 corresponding to the switch points or marked points on ∂R , obtained by Lemma 6.18.

Let $\tilde{\nu}_i$ be the total lift of ν_i to \tilde{C}_i . Then the leaves of ν_i are rails of T_i . Since T_i is ϵ -straight with sufficiently small $\epsilon > 0$, every rail l_i of \tilde{T}_i has curvature at most ϵ . Thus, for every $\delta > 0$, by taking sufficiently small $\epsilon > 0$, the smooth curve $\beta_i|_{l_i}$ is $(1 + \delta)$ -bilipschitz for all rails l_i of $\tilde{T}_i (= \tilde{T}_i(\epsilon))$. Then we have an analogue of Proposition 6.21 for $\tilde{\mathcal{T}}_i$:

Proposition 6.24. *For every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then, for every rail m_i of $\tilde{\mathcal{T}}_i$ passing through a branch \mathcal{R}_i , letting l_i be the corresponding rail of \tilde{T}_i (passing through R_i), we have*

- (i) $f_i|_{m_i}$ is a simple arc on $\hat{\mathbb{C}}$ connecting the end points of the bilipschitz curve $\beta_i(l_i) \subset \mathbb{H}^3$ and it intersects c_1 and c_2 transversally in a single point; thus $f^{-1}(\mathcal{A}) \cap m_i$ is a single arc in m_i ,
- (ii) $f_i^{-1}(\mathcal{A}) \cap m_i$ is δ -close to the rail $\mathcal{R}_i \cap m_i$ of \mathcal{R}_i with the Hausdorff metric, and
- (iii) if an endpoint of $\mathcal{R}_i \cap m_i$ is a switch point of $\tilde{\mathcal{T}}_i$, then the δ -deformation of $\mathcal{R}_i \cap m_i$ to $f_i^{-1}(\mathcal{A}) \cap m_i$, given by (ii), preserves this switch point.

Proof. (Case 1.) Let \mathcal{R}_i be a branch of $\tilde{\mathcal{T}}_i$. For every leaf m_i of $\tilde{\nu}_i$ passing through \mathcal{R}_i , let l_i be the leaf of $\tilde{\lambda}_i$ corresponding to m_i . Then l_i intersects the branch R_i of \tilde{T}_i corresponding to \mathcal{R}_i . Then $f|_{m_i}$ is a circular arc on $\hat{\mathbb{C}}$ connecting the endpoints of $\beta_i(l_i)$. By Lemma 6.15, for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then, if l_i and l are rails of \tilde{T}_i and \tilde{T} passing through corresponding branches R_i and R of \tilde{T}_i and \tilde{T} , respectively, then $\beta_i(l_i \cap R_i)$ and $\beta(l \cap R)$

are geodesic segments, in \mathbb{H}^3 , of length at least K that are δ -close with the Hausdorff metric. We have seen that, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then for every leaf m of $\tilde{\lambda}_\infty$ passing through R , $\beta(l)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ at an angle δ -close to $\pi/2$, and $\beta^{-1}(\text{Conv}(\mathcal{A})) \cap m$ is δ -close to $m \cap R$ (Lemma 6.18 and Proposition 6.21). Since $\beta(l \cap R)$ and $\beta_i(l_i \cap R_i)$ are sufficiently close, we have

Claim 6.25. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small then, for every leaf m_i of $\tilde{\lambda}_i$ passing through R_i , the geodesic $\beta_i(l_i)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ δ -orthogonally, and $\beta^{-1}(\text{Conv}(\mathcal{A})) \cap m_i$ is δ -close to $m_i \cap \mathcal{R}_i$ in the Hausdorff metric.*

Therefore, similarly to the proof of Proposition 6.21 (Case 1), we can prove that, if i is sufficiently large, then, every leaf m_i of $\tilde{\lambda}_i$ passing through a branch \mathcal{R}_i satisfies (i) and (ii). Since m_i is a leaf of $\tilde{\nu}_i$ and $\tilde{\nu}_i$ is contained in the interior of $\tilde{\mathcal{T}}_i$, endpoints of the rail $m_i \cap \mathcal{R}_i$ can not be a switch point of $\tilde{\mathcal{T}}_i$.

(Case 2.) Suppose that m_i is a rail of $\tilde{\mathcal{T}}_i$ intersecting a branch \mathcal{R}_i of $\tilde{\mathcal{T}}_i$. Then let l_i be the corresponding rail of $\tilde{\mathcal{T}}_i$ intersecting R_i . Pick a leaf l'_i of $\tilde{\lambda}_i$ passing through R_i that is closest to l_i in R_i . Then there is a component X_i of $\mathbb{H}^2 \setminus \tilde{\lambda}_i$ bounded by l'_i and containing l_i . Then β_i isometrically embeds X_i into a totally geodesic hyperplane in \mathbb{H}^3 . For every $\delta > 0$, if $\tilde{\mathcal{T}}_i$ is ϵ -slim with sufficiently small $\epsilon > 0$, then $l_i \cap R_i$ and $l'_i \cap R_i$ are δ -close for all R_i and all corresponding l_i and l'_i as above. Therefore, using Claim 6.25 and imitating the proof of Proposition 6.21 (Case 2), we can show that, every rail m_i of $\tilde{\mathcal{T}}_i$ passing through a branch \mathcal{R}_i of $\tilde{\mathcal{T}}_i$ satisfies (i) and (ii).

Last we show (iii). Suppose that m_i passes through a switch point \mathbf{p}_i on $\partial \mathcal{R}_i$. Then let \mathbf{p} be the corresponding switch point on $\partial \mathcal{R}$. Then $f(\mathbf{p})$ is on the boundary component of \mathcal{A} corresponding to $\tilde{\kappa}(\mathbf{p})$ (see Lemma 6.18). In addition, by Proposition 6.11 (iii), $f_i(\mathbf{p}_i) = f(\mathbf{p})$. Therefore the switch point \mathbf{p}_i is preserved under the δ -deformation of $m_i \cap \mathcal{R}_i$ to $m_i \cap f_i^{-1}(\mathcal{A})$. 6.24

Fix small $\delta > 0$. Then assume that $i \in \mathbb{N}$ is sufficiently large so that the conclusions of Proposition 6.24 are satisfied for this δ . Let \mathcal{R}_i be a branch of $\tilde{\mathcal{T}}_i$, and let \mathcal{R} be the corresponding branch of \mathcal{T}_∞ . Then $\mathcal{R} \subset \tilde{C}$ is a quadrangle supported on the round cylinder \mathcal{A} . We set $\mathcal{R}_i = \cup(m_i \cap \mathcal{R}_i)$, where the union runs over all rail m_i of $\tilde{\mathcal{T}}_i$ passing through \mathcal{R}_i . Then let $\alpha_{m_i} = m_i \cap \mathcal{R}_i$ and let $\alpha'_{m_i} = m_i \cap f^{-1}(\mathcal{A})$. Then, by Proposition 6.24, α'_{m_i} and α_{m_i} are segments in m_i that are δ -close, and α'_{m_i} is supported on \mathcal{A} . Therefore, similarly to Corollary 6.22, \mathcal{R}'_i is

a quadrangle supported on the round cylinder \mathcal{A} . By Proposition 6.11 (iii), $\partial\mathcal{T}_\infty$ and $\partial\mathcal{T}_i$ are isomorphic via the developing maps. Therefore, we see that \mathcal{R}'_i and \mathcal{R} have the same pair of supporting arcs as well. By applying this δ -deformation to all branches of $\tilde{\mathcal{T}}_i$ simultaneously and descending to \mathcal{T}_i , we obtain a desired deformation. 6.23

6.8. Estimate of multiarcs for grafting by transversal measure.

We have proved Proposition 6.10 except (II - ii) and (II - iii), which we prove in this section. Recalling Proposition 6.11, let B be a branch of the traintrack T on S . Set $R_i = \phi_i(B)$ and $R = \phi(B)$, which are the corresponding branches of \tilde{T}_i and \tilde{T}_∞ , respectively. Let \mathcal{R}_i and \mathcal{R} denote the branches of $\tilde{\mathcal{T}}_i$ and $\tilde{\mathcal{T}}_\infty$ corresponding to R_i and R , respectively, obtained by Proposition 6.17 and Proposition 6.23. Then, \mathcal{R}_i and \mathcal{R} are projective structures on a quadrangle with the same support. Thus \mathcal{R}_i and \mathcal{R} are equipped with the canonical foliations induced by the canonical foliation $\mathcal{F}_\mathcal{A}$ on \mathcal{A} , and we assume that the ties of \mathcal{R}_i and \mathcal{R} are the leaves of those foliations by perturbing ties. By Lemma 6.5, we have either $\mathcal{R} = Gr_M(\mathcal{R}_i)$ or $\mathcal{R}_i = Gr_M(\mathcal{R})$ for some admissible multiarc M (II -ii). We are left to show

Proposition 6.26. *For every $\delta > 0$, if i is sufficiently large and $\epsilon > 0$ is sufficiently small, then, for all corresponding branches \mathcal{R}_i and \mathcal{R} of $\tilde{\mathcal{T}}_i$ and $\tilde{\mathcal{T}}_\infty$, we have $\mathcal{R}_i = Gr_M(\mathcal{R})$ for some admissible multiarc M on \mathcal{R} , and $(\tilde{\mu}_i(R_i) - \tilde{\mu}(R))/2\pi$ is δ -close to the number of arcs of M .*

The rest of this section is the proof of this proposition. Pick a leaf h of $\mathcal{F}_\mathcal{A}$, which is a round circle on $\hat{\mathbb{C}}$. Since we have assumed that the ties of \mathcal{R} are induced by $\mathcal{F}_\mathcal{A}$, there is a unique tie a of \mathcal{R} such that f immerses a into h . Set $\alpha = f|_a$. Similarly let a_i be the tie of \mathcal{R}_i such that f_i immerses a_i into h . Set $\alpha_i = f_i|_{a_i}$.

As in §6.1, let g be the geodesic in \mathbb{H}^3 orthogonal to \mathcal{A} . Then, in particular, g is orthogonal to h . We identify $\hat{\mathbb{C}}$ with \mathbb{S}^2 , by an element of $\mathrm{PSL}(2, \mathbb{C})$, so that $h \subset \hat{\mathbb{C}}$ is the equator of \mathbb{S}^2 and g connects the north pole and the south pole of \mathbb{S}^2 . We assume that \mathbb{S}^2 is the unit sphere, and then the spherical metric on \mathbb{S}^2 induces a metric on h , so that $\mathrm{length}_h(h) = 2\pi$. Then $\mathrm{length}_h(\alpha)$ and $\mathrm{length}_h(\alpha_i)$ denote the lengths of α and α_i , respectively, with respect to this metric on h .

Recall that, given any $\epsilon > 0$, we can assume that T and T_i , with sufficiently large i , are ϵ -straight and slim (Proposition 6.11). Then

Proposition 6.27. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then we have*

- (i) $|length_h(\alpha) - \tilde{\mu}(R)| < \delta$ for every branch R of \tilde{T}_∞ and every tie a of the branch \mathcal{R} of \tilde{T}_∞ corresponding to R , and
- (ii) if $i \in \mathbb{N}$ is sufficiently large, we have, $|length_h(\alpha_i) - \tilde{\mu}_i(R_i)| < \delta$ for the branch R_i of \tilde{T}_i corresponding to R .

Proof of Proposition 6.26 with Proposition 6.27 assumed. By Lemma 6.16, if i is sufficiently large, then we can assume that $\tilde{\mu}_i(R_i) - \tilde{\mu}(R) > 0$ for all corresponding branches R_i and R of \tilde{T}_i and \tilde{T}_∞ , respectively. Pick corresponding ties a and a_i of \mathcal{R} and \mathcal{R}_i , respectively, that map to the same round circle h on $\hat{\mathbb{C}}$ via the developing maps f and f_i . Then, by Proposition 6.27, for every $\delta > 0$, if $\epsilon > 0$ and sufficiently small and $i \in \mathbb{N}$ is sufficiently large, then $length_h(\alpha_i)$ is δ -close to $\mu_i(R_i)$ and $length_h(\alpha)$ is δ -close to $\tilde{\mu}(R)$ for all corresponding R and R_i . Then, since \mathcal{R} and \mathcal{R}_i have the same support (Proposition 6.23 (ii)), $length_h(\alpha_i) - length_h(\alpha)$ is a 2π -multiple. By Lemma 6.5, we have $\mathcal{R}_i = Gr_M(\mathcal{R})$ for some admissible multiarc M on \mathcal{R} . By the definition of grafting, we see that the number of arcs of M is exactly $(length_h(\alpha_i) - length_h(\alpha))/2\pi$. Therefore $(\tilde{\mu}_i(R_i) - \tilde{\mu}(R))/2\pi$ is 2δ -close to the number of arcs of M . 6.26

Proof of Proposition 6.27. We prove (i). The prove of (ii) is similar, since we have a convergence $\beta_i \rightarrow \beta$ (Theorem 5.4). Throughout this proof, $\delta > 0$ is a sufficiently small number, which may be appropriately replaced by a smaller number by taking smaller $\epsilon > 0$, where T_∞ is ϵ -straight and slim; however δ does *not* depend on the choice of the branch R of \tilde{T}_∞ .

Recall that we have a measured geodesic lamination $L = (\lambda, \mu)$ on τ and that the geodesic lamination λ is a sublamination of λ_∞ . Set $L_\infty = (\lambda_\infty, \mu)$, which fully embeds into \tilde{T}_∞ (Proposition 6.11). For an arbitrary branch R of T_∞ , let $I \in \mathcal{ML}(\mathbb{H}^2)$ be the intersection $I(\tilde{L}, R)$ of \tilde{L} and R (see §3.3). Set $I = (\lambda_I, \mu_I)$, where $\lambda_I \in \mathcal{GL}(\mathbb{H}^2)$ and $\mu_I = \tilde{\mu}|_{\lambda_I}$. Accordingly, let $\beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ denote the bending map induced by I . The total measure of I is $\tilde{\mu}(R) < \infty$, and thus, for every geodesic segment s on \mathbb{H}^2 transversal to I , we have $\mu_I(s) \leq \tilde{\mu}(R) \in \mathbb{R}_{\geq 0}$. Thus $\beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ continuously extends to $\partial\mathbb{H}^2 \cong \mathbb{S}^1$.

Since each component of $\mathbb{H}^2 \setminus \lambda_I$ is bounded by at most 2 leaves of λ_I , we can extend the geodesic lamination λ_I to a geodesic foliation F_I on \mathbb{H}^2 (so that the dual tree of F_I is a copy of \mathbb{R}). Furthermore, since T_∞ is ϵ -slim, we can assume that, for every $\delta > 0$, if $0 < \epsilon < \delta$ is sufficiently small, then if a leaf of F_I intersects a tie of R at some point, then the intersection angle is δ -close to $\pi/2$.

Let C_I denote the projective structure on the open disk \mathbb{D}^2 associated with the measured lamination I on \mathbb{H}^2 . Since R is connected and I is a sublamination of \tilde{L}_∞ , then C_I canonically embeds into \hat{C} (see [2]). In particular, since $I = \tilde{L}_\infty$ on R , then \mathcal{R} canonically embeds into C_I . Let $f_I: \mathbb{D}^2 \rightarrow \hat{C}$ denote the developing map of C_I , and let \mathcal{F}_I denote the canonical foliation on C_I corresponding to F_I via the collapsing map $\kappa_I: C_I \rightarrow \mathbb{H}^2$. Then the dual tree of \mathcal{F}_I is also a copy of \mathbb{R} . Since C_I is Gromov-hyperbolic with Thurston's metric, $\partial C_I \cong \mathbb{S}^1$. We then pick a continuous projection $\Phi: C_I \rightarrow \partial C_I$ along leaves of \mathcal{F}_I , i.e. for each point $x \in C_I$, $\Phi(x)$ is an endpoint of the leaf of \mathcal{F}_I containing x . Since T_∞ is ϵ -straight, each tie a of \mathcal{R} is transversal to the foliation \mathcal{F}_I . Thus Φ projects a into a simple arc in ∂C_I (note that, if a intersects a Euclidean part of C_I , then $\Phi|_a$ is *not* injective). Let b denote the simple arc $\Phi(a)$.

Since $\beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ extends to $\partial \mathbb{H}^2$, the collapsing map $\kappa_I: C_I \rightarrow \mathbb{H}^2$ continuously extends to a homeomorphism from ∂C_I to $\partial \mathbb{H}^2$, so that $f_I = \beta_I \circ \kappa_I$ on ∂C_I . In particular, κ_I homeomorphically takes b onto its image $\kappa_I(b) \subset \partial \mathbb{H}^2$. Then let $\beta_b: b \rightarrow \hat{C}$ denote the restriction of β_I to b .

The transversal measure μ_I of I is defined for arcs in \mathbb{H}^2 transversal to I . Since the total measure of μ_I is finite, μ_I continuously extends to arcs contained in $\partial \mathbb{H}^2$. Thus I continuously extends to a measured lamination on $\partial \mathbb{H}^2$ (whose leaves have dimension 0), and thus, in particular to b .

Recall that h is a leaf of the foliation \mathcal{F}_A on the round cylinder $\mathcal{A} \subset \hat{C}$ supporting $\mathcal{R} \subset \tilde{C}$ and that the geodesic $g \subset \mathbb{H}^3$ is orthogonal to \mathcal{A} . By Lemma 6.18, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then for all leaves l of F_I intersecting R , the geodesic segments $\beta_I(l \cap R)$ and $g \cap \text{Conv}(\mathcal{A})$ are δ -close with the Hausdorff metric. Thus, assuming $\epsilon > 0$ is sufficiently small, $Im(\beta_b)$ is contained in a small neighborhood of an ideal endpoint O of g . In particular, $Im(\beta_b)$ is contained in a component D of $\hat{C} \setminus h$, which is a round disk. In the following, D is equipped either with the spherical metric, regarded as a upper hemisphere (so that O is the north pole and h is the equator), or with the hyperbolic metric, regarded as the Poincare disk. Then, since $\beta_b: b \rightarrow \hat{C}$ is the boundary of the bending map $\beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3$, we see that

Lemma 6.28. *The curve $\beta_b: b \rightarrow D$ is smooth except at the end points of leaves of I , and b is bent at those endpoints by angles corresponding to μ_I . In addition, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then*

the curvature of β_b is less than $\delta > 0$ at every smooth point (with either metric on D).

By the definition of $\Phi: C_I \rightarrow \partial C_I$, for each $x \in a \subset C_I$, there is a unique ray contained in a leaf l of \mathcal{F}_I that connects x to $\Phi(x)$. Then f_I homeomorphically takes this ray to a circular arc embedded in $\hat{\mathbb{C}}$ that connects the point $f_I(x)$ to the point $f_I(\Phi(x)) = \beta_b(\Phi(x))$. Let $r_x: [0, 1] \rightarrow D \subset \mathbb{S}^2$ denote this circular arc with $r_x(0) = f_I(\Phi(x))$ and $r_x(1) = f_I(x)$. Then, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then r_x intersects the round circle h , δ -orthogonally, at the end point $f_I(x)$, and $\gamma_x(0)$ is δ -close to the center O .

There, is a unique maximal ball in C_I whose core contains x . By the definition of a maximal ball, its f_I -image is a round open ball in $\hat{\mathbb{C}}$, which we denote by D_x . Then, ∂D_x bounds a totally geodesic hyperplane in \mathbb{H}^3 , and the circular arc $f_I(l) \subset \hat{\mathbb{C}}$ orthogonally projects to the geodesic $\beta_I \circ \kappa_I(l)$ contained in this hyperplane. Thus $r_x \subset f_I(l)$ orthogonally intersects ∂D_x at the endpoint $r_x(0)$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, since $\kappa_I(l)$ is a leaf of F_I intersecting R , the geodesic $\beta_I(\kappa_I(l))$ is δ -orthogonal to the totally geodesic hyperplane $\text{Conv}(h)$ in \mathbb{H}^3 (Lemma 6.18 (iii)). Therefore we can in addition assume that the curvature of $r_x: [0, 1] \rightarrow D \subset \mathbb{S}^2$ is less than δ with respect to the spherical metric on D .

Let $\mathbf{r}_x: [0, 1] \rightarrow D$ be the hyperbolic geodesic ray in $D \cong \mathbb{H}^2$ connecting the endpoints of r_x . Then, since $r_x(0)$ is δ -close to the center O of D , for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then \mathbf{r}_x and r_x are δ -close (with either metric on D) for all $x \in a$.

Lemma 6.29. *For every $x \in a$, there exists a small neighborhood U_x of x in a , such that if $y \in U_x$, then $\mathbf{r}_x|_{(0, 1]}$ and $\mathbf{r}_y|_{(0, 1]}$ are disjoint.*

Proof. Let $x \in a$. Assume that $y \in a$ is sufficiently close to x . If $\mathbf{r}_x(0) = \mathbf{r}_y(0)$, then we have $\mathbf{r}_x|_{(0, 1]} \cap \mathbf{r}_y|_{(0, 1]} = \emptyset$ since $f_I(x) \neq f_I(y) \in \partial D$. Thus we can assume that $\mathbf{r}_x(0) \neq \mathbf{r}_y(0)$. Then, since β_b is a further restriction of $\beta_I|_{\partial \mathbb{H}^2}$ and x and y are sufficiently close, the round circles ∂D_x and ∂D_y on $\hat{\mathbb{C}}$ intersects at an angle sufficiently close to 0. Let l be the geodesic in $D \cong \mathbb{H}^2$ connecting $\mathbf{r}_x(0)$ and $\mathbf{r}_y(0)$. Then, since r_x and r_y are orthogonal to ∂D_x and ∂D_y , respectively, they are disjoint circular arcs contained in the closure of a component of $D \setminus l$. Since \mathbf{r}_x and \mathbf{r}_y share endpoints with r_x and r_y , respectively, \mathbf{r}_x and \mathbf{r}_y are disjoint geodesics rays in the closure of the component of $D \setminus l$. \square

For each $x \in a$, let $\gamma_x: [0, 1] \rightarrow D$ denote the circular arc on $D \subset \mathbb{S}^2$ connecting the center O to $r_x(1) = f_I(x)$. Then γ_x intersects $h = \partial D$ orthogonally at $\gamma_x(1)$. Identify a with the closed interval $[0, \text{length}_h(a)]$

so that $f_I|_a: a \rightarrow h \cong \mathbb{S}^1$ is an isometric immersion. Then, as $x \in a \cong [0, \text{length}_h(a)]$ increases, the circular arc $\gamma_x: [0, 1] \rightarrow D$ ($x \in [0, 1]$) continuously rotates in one direction by the rotation of D fixing the center $O = \gamma_x(0)$. Define $\gamma: a \times [0, 1] \rightarrow D$ by $\gamma(x, t) = \gamma_x(t)$. Then $\gamma(x, 1) = f_I(x)$ for all $x \in a$. Thus $\gamma|_{a \times (0, 1]}$ is an immersion, and, more precisely, γ is a fan immersed into D , where the vertex of the fan is O and the angle of the fan (at the vertex) is $\text{length}_h(\alpha)$. Let $\hat{F} \cong a \times [0, 1]$ denote the domain of γ and equip \hat{F} with the pull back metric of the spherical metric on D via γ . Then we have

$$(4) \quad \text{Area}(\hat{F}) = \text{Area}_{\mathbb{S}^2}(D) \cdot (\text{length}_h(\alpha)/2\pi) = \text{length}_h(\alpha).$$

Similarly define $\mathbf{r}: a \times [0, 1] \rightarrow D$ by $\mathbf{r}(x, t) = \mathbf{r}_x(t)$. Then, by Lemma 6.29, $\mathbf{r}|_{a \times (0, 1]}$ is an immersion. Clearly $\mathbf{r}|_{a \times \{0\}} \rightarrow D = \beta_b \circ \Phi: a \rightarrow D$ and $\mathbf{r}|_{a \times \{1\}} = \alpha: a \rightarrow \partial D$. Let F be $\text{Domain}(\mathbf{r}) = a \times [0, 1]$ with the pull-back metric of the spherical metric D via \mathbf{r} . Then the four boundary edges of F correspond to α , β_b , \mathbf{r}_0 , and \mathbf{r}_1 . Applying the Gauss-Bonnet theorem to F , we have

$$\text{Area}(F) + \int_{\partial F} k ds + \Sigma \theta_p = 2\pi \cdot \chi(F),$$

where k is the curvature along ∂F and θ_p are the exterior angles at non-smooth points p of ∂F , including infinitesimal bending angles of β_b corresponding to μ_I (Lemma 6.28). Then, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then the third term $\Sigma \theta_p$ is δ -close to $-\tilde{\mu}(R) + 2\pi$: It is “ δ -close” since \mathbf{r}_0 and \mathbf{r}_1 are almost orthogonal to ∂D_0 and $\partial D_{\text{length}(a)}$, respectively. Next consider the second term $\int_{\partial F} k ds = \int_{\beta_b \cup \mathbf{r}_0 \cup \mathbf{r}_1 \cup \alpha} k ds$. Since ∂D is a geodesic loop on \mathbb{S}^2 , $\int_{\alpha} k ds = 0$. Since \mathbf{r}_x is sufficiently close to r_x for each $x \in a$, we can assume that, with respect the spherical metric on $D \subset \mathbb{S}^2$, \mathbf{r}_0 and \mathbf{r}_1 have total curvature less than δ . Since \tilde{T}_∞ is ϵ -slim and $\text{Im}(\beta_b)$ is contained in a δ -neighborhood of O , $\beta_b: b \rightarrow a$ is sufficiently short and it has sufficiently small curvature at smooth points. Therefore, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $|\int_{\partial F} k ds| < \delta$. Since F is topologically a disk, $\chi(F) = 1$. Thus we have

Lemma 6.30. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then,*

$$-\delta < \text{Area}(F) - \tilde{\mu}(R) < \delta.$$

Next we have

Lemma 6.31. *(i) For every subinterval d of $[0, \text{length}_h(\alpha)] \cong a$ of length less than 2π , γ embeds $d \times (0, 1]$ into D . (ii) For every $\delta > 0$, if $\epsilon > 0$ sufficiently small, then \mathbf{r} embeds $d \times (0, 1]$ into D for every subinterval d of $[0, \text{length}_{\mathbb{S}^2}(\alpha)] \cong a$ of length less than $\pi - \delta$.*

Proof. (i) If d is a subinterval of $[0, \text{length}_h(a)]$, the restriction of γ to $d \times [0, 1]$ is a fan centered at O such that $\text{length}(d)$ is equal to the length of the circular arc of the fan. Thus, if $d < 2\pi$, then $\gamma: d \times (0, 1]$ is an embedding. (ii) Let $\mathcal{I} = (\lambda_{\mathcal{I}}, \mu_{\mathcal{I}})$ be the canonical measured lamination on C_I corresponding to I via the collapsing map $\kappa_I: C_I \rightarrow \mathbb{H}^2$. Since β_b is the restriction on the (extended) bending map β and it is a short curve contained in a small neighborhood of the center O , for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\mu_{\mathcal{I}}(d)$ is δ -close to $\text{length}_h(f|d)$ for every subinterval d of a with bounded length (this “bounded” assumption is more important for Proposition 6.27 (ii)). In particular, assuming $\epsilon > 0$ is sufficiently small, if $\text{length}(d) < \pi - \delta$, then $\mu_{\mathcal{I}}(d) < \pi - \delta/2 < \pi$. Then, for such d , $\beta_b|_{\Psi(d)}$ is injective by the Gauss-Bonnet theorem, since β_b has sufficiently small curvature at smooth points and $\beta_b|_{\Psi(d)}$ is sufficiently short. Thus, since \mathbf{r}_x are δ -orthogonal to β_b at each $x \in a$, we see that $\mathbf{r}_x|_{(0, 1]}$ ($x \in d$) are pairwise disjoint. \square

By (4) and Lemma 6.30, to prove Proposition 6.27 (i), it suffices to show

Proposition 6.32. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then*

$$-\delta < \text{Area}(F) - \text{Area}(\hat{F}) < \delta$$

for every branch R of \tilde{T}_{∞} and every tie a of the branch \mathcal{R} of \tilde{T}_{∞} corresponding to R .

Proof. Fix sufficiently small $\delta > 0$. For each $y \in b$, let $q_y: [0, 1] \rightarrow D$ be the geodesic segment connecting O to $\beta_b(y)$. Since $\text{Im}(\beta_b)$ is contained in a sufficiently small neighborhood of O , we can assume that $\text{length}_{\mathbb{S}^2}(q_y)$ is less than δ for all $y \in b$. Define $q: b \times [0, 1] \rightarrow D$ by $q(y, t) = q_y(t)$. Let Q be $b \times [0, 1]$ equipped with the 2-form obtained by pulling back the spherical Riemannian metric of D via q . Since $\beta_b: b \rightarrow D \subset \mathbb{S}^2$ is locally injective, we assign a metric on b by the arc length of β_b . Then $0 \leq \text{Area}(Q) < \int_{y \in b} \text{length}(q_y) dy$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\text{Area}(Q) < \delta$, since $\text{length}(q_y)$ and $\text{length}(\beta_b)$ are sufficiently small.

Let v_0 and v_1 denote the endpoints of $a \cong [0, \text{length}_h(\alpha)]$ corresponding to 0 and $\text{length}(\alpha)$, respectively. For each $k = 0, 1$, let E_k be the region in D bounded by \mathbf{r}_{v_k} , γ_{v_k} and q_{v_k} . Since q_{v_k} has length less than δ , by taking smaller $\epsilon > 0$ if necessarily, we can assume that $\text{Area}_{\mathbb{S}^2}(E_{v_k})$ ($k = 0, 1$) are less than δ . Thus, to prove Proposition 6.32 (and therefore Proposition 6.27), it suffices to show:

Claim 6.33. $|\text{Area}(F) - \text{Area}(\hat{F})| < \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1)$.


$$Area(\hat{F}) < Area(Q) + Area(E_0) + Area(E_1) + Area(F).$$

- if $\mathbf{r}_x \subset \gamma_x$ or $\mathbf{r}_x \supset \gamma_x$, then $\xi(x) = x$,
- otherwise, letting d_x be the closed subinterval of a with the endpoints x and $\xi(x)$, d_x is the largest subinterval containing x and satisfying $\mathbf{r}_x \cap \gamma_y \neq \emptyset$ for all $y \in d_x$.

- (i) \mathbf{r}_y with $y \in d_x$ contains O if and only if $y = \xi(x)$ (Figure 6),
- (ii) the point $\mathbf{r}_y(0) = \beta_b \circ \Phi(y)$ with $y \in d_x$ intersects γ_x if and only if $y = \xi(x)$ (Figure 7), or
- (iii) $\xi(x)$ is an endpoint of a (Figure 8).

We also have

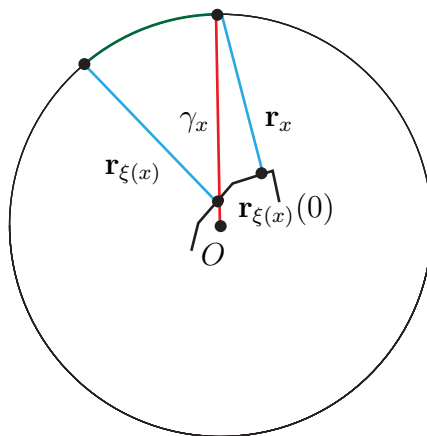


FIGURE 7.

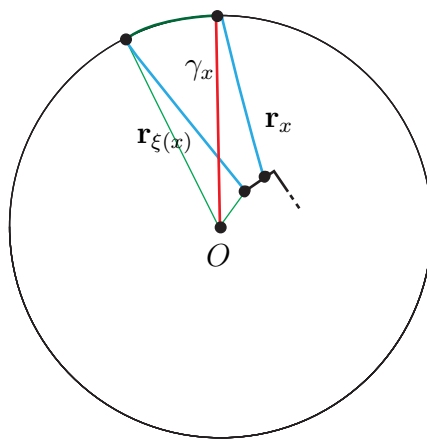


FIGURE 8.

Lemma 6.35. *For all $x \in a$,*

$$\text{length}(d_x) < \pi.$$

Moreover, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then if $x \in a$ is of Type (ii), then

$$\text{length}(d_x) < \pi/2 + \delta.$$

Proof. For each $x \in a$, let l_x be the geodesic l_x in $D \cong \mathbb{H}^2$ containing γ_x . Then l_x divides D into two semicircles. For each Type of (i), (ii), (iii), we see that f_I embeds d_x into one of those semicircles. Thus $\text{length}(d_x) < \pi$.

Suppose that x is of Type (ii). Then, since $\gamma_x \cap \mathbf{r}_y \neq \emptyset$ for all $y \in d_x$ and $\mathbf{r}_{\xi(x)}$ is δ -orthogonal to β_b at the point $\mathbf{r}_{\xi(x)}(0) \in \gamma_x$, the angle between $\mathbf{r}_{\xi(x)}$ and γ_x is less than $\pi/2 + \delta$ at $\mathbf{r}_{\xi(x)}(0)$. Since $\mathbf{r}_{\xi(x)}(0)$ is sufficiently close to O , thus $\text{length}(d_x) < \pi/2 + \delta$. \square

Observe that the correspondence between points x in a and Types (i), (ii) (iii) changes only at finitely many points on a : The type may change between (i) and (ii) when β_b is tangent to γ_x ; Type (iii) happens exactly in a (possibly empty) subinterval of a starting at an end point of a and, at the other endpoint of this subinterval, the type changes from (iii) to (ii).

Suppose that $x \in a$ is of Type (i). Then $\gamma_x: [0, 1] \rightarrow D$ is embedded in $\mathbf{r}(d_x \times [0, 1])$. Moreover, since $\beta_b \circ \Phi|_{d_x}$ is disjoint from γ_x , we see that γ_x factors through \mathbf{r} via an unique isometric embedding ι_x of $[0, 1]$ into $d_x \times [0, 1] = \text{Domain}(\mathbf{r}) = F$.

Suppose that $x \in a$ is of Type (ii). Then the points $\beta_b \circ \Phi \circ \xi(x)$ divides γ_x into two segment: One segment γ_x^- is $q(\Phi \circ \xi(x), *)$ and the other segment γ_x^+ is contained in $\mathbf{r}(d_x \times [0, 1])$. By Lemma 6.35, we can assume that $d_x < 2\pi/3$. Thus, by Lemma 6.31, $\mathbf{r}|_{d_x \times (0, 1]}$ is an embedding. Thus, γ_x^+ factors through \mathbf{r} via an isometric embedding of the domain of γ_x^+ into $d_x \times [0, 1] \subset F$. Clearly γ_x^- factors through q via the identification of γ_x^- and $q(\Phi \circ \xi(x), *)$.

Suppose that $x \in a$ is of Type (iii). Then γ_x is a curve embedded in $\mathbf{r}(d_x \times [0, 1]) \cup E_k$ with $k \in \{0, 1\}$ satisfying $v_k = \xi(x)$. Thus, similarly, $\gamma_x: [0, 1] \rightarrow D$ factors through an isometrically embedding of the domain of γ_x into $d_x \times [0, 1] \sqcup E_k$ via \mathbf{r} , if we remove, from the domain of γ_x , the single point corresponding to the intersection of γ_x and $\mathbf{r}_{\xi(x)}$. Thus, for all $x \in a$, let ι_x be the above isometric embedding of (the domain of) γ_x possibly minus a single point into $F \sqcup E_0 \sqcup E_1 \sqcup Q$. Then we see that ι_x changes continuously in $x \in a$. Let X be the set of the points $(x, t) \in \hat{F} \cong a \times [0, 1]$ that are removed from the domain of γ_x to define ι_x . Then, since ι_x is defined $[0, 1]$ minus at most a single point for each $x \in a$, X has measure zero in \hat{F} . In addition, since the configuration type (i.e. (i), (ii), (iii)) changes at only finitely many $x \in a$, X decompose \hat{F} into finitely many (connected) components. We then define $\iota: \hat{F} \setminus X \rightarrow F \sqcup E_0 \sqcup E_1 \sqcup Q$ by $\iota|_{\gamma_x} = \iota_x$ for all $x \in a$. Since ι_x changes continuously in x , then ι is a continuous. Moreover ι is an isometric immersion (up to measure zero), since the Riemannian

metrics on \hat{F}, F, E_0, E_1, Q are all induced by the spherical metric on $D \subset \mathbb{S}^2$. Thus the only thing left to show is that ι is almost everywhere injective:

Claim 6.36. *For almost all different $x, y \in a$, ι_x and ι_y have disjoint images, up to measure zero.*

Proof. We can assume that $f(x) = f(y)$: Otherwise γ_x and γ_y are disjoint (except at the point O) and thus so are $\text{Im}(\iota_x)$ and $\text{Im}(\iota_y)$. Then $\gamma_x = \gamma_y$ and $|x - y| \geq 2\pi$.

First we show that $\text{Im}(\iota_x) \cap F$ and $\text{Im}(\iota_y) \cap F$ are disjoint. Then $\text{Im}(\iota_x) \cap F$ and $\text{Im}(\iota_x) \cap F$ are a single arc contained in $F \cong a \times [0, 1]$ with their endpoints at $(x, 1)$ and $(y, 1)$, respectively. We have seen that \mathbf{r} immerses $a \times (0, 1]$ into D . Therefore, since $\mathbf{r} = \gamma \circ \iota$ in $\iota^{-1}(F)$ and $\gamma_x = \gamma_y$, if ι_x and ι_y intersect at a point in $a \times (0, 1] \subset F$, then $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap F$ contains an arc whose endpoint is on $a \times \{1\}$. This implies that $x = y$, which is a contradiction.

We next show that $\text{Im}(\iota_x) \cap Q$ and $\text{Im}(\iota_y) \cap Q$ are disjoint up to measure zero. Then, without loss of generality, we can assume that x and y are of Type (ii) and that $x < y$. In addition, by lemma 6.35, we can assume that $\text{length}(d_x) < \pi/2 + \delta$ and $\text{length}(d_y) < \pi/2 + \delta$ with sufficiently small δ . Therefore, since $|x - y| \geq 2\pi$, we must have $d_x \cap d_y = \emptyset$. In particular, $\xi(x) < \xi(y)$, and then, the length of the interval $[\xi(x), \xi(y)]$ is more than $\pi - 2\delta > 0$. Suppose that $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap Q \neq \emptyset$; then we must have $\Phi \circ \xi(x) = \Phi \circ \xi(y)$. Since the non-injectivity of $\Phi|_a: a \rightarrow b$ corresponds the non-injectivity of the collapsing maps $\kappa_I: C_I \rightarrow \mathbb{H}^2$, for all $z_1, z_2 \in a = [0, \text{length}_h(\alpha)]$ with $z_1 < z_2$, we have $\Phi(z_1) \leq \Phi(z_2)$. Then Φ must collapse the closed interval $[\xi(x), \xi(y)]$ to a single point, so that $\mathbf{r}_z(0)$ is constant for all $z \in [\xi(x), \xi(y)]$. Then we can observe that, for all but finitely many points in (x, y) are of Type (i) and those finitely many exceptional points z are of Type (ii) with $\mathbf{r}_z \subset \gamma_z$. Thus $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap Q \neq \emptyset$ for only finitely many distinct pairs of points, (x, y) , of a .

We last show that, for each $k = 0, 1$, if $x, y \in a$ are distinct, then $\text{Im}(\iota_x) \cap E_k$ and $\text{Im}(\iota_y) \cap E_k$ are disjoint. It suffices to show for $k = 0$ by the symmetry. Recall that $\xi(x) = 0$ if and only if x in a (possibly empty) interval $[0, w] \subset [0, \text{length}(\alpha)] \cong a$ with some w . Then, since E_0 is an (ideal) geodesic triangle with a vertex at O and $(\gamma_z \setminus O) \cap E_0 \neq \emptyset$ for all $z \in [0, w]$, we see that $w < \pi$. Therefore $\gamma_x \setminus O$ are disjoint for different $x \in [0, w]$. Therefore, $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap E_0$ is at most a single point, which corresponds to O , for all different $x, y \in a$. \square

Next we show

$$\text{Area}(F) < \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1) + \text{Area}(\hat{F})$$

to complete the proof of Claim 6.33. For this, we similarly construct an almost-everywhere-defined map $\iota: F \rightarrow \hat{F} \sqcup Q \sqcup E_0 \sqcup E_1$ that is a piecewise isometric-embedding but, in contrast with the previous case, *not* necessarily an embedding. Nevertheless, this map will help us to compare the areas.

We see that there is a unique continuous function $\xi: a \rightarrow a$ such that

- if $\mathbf{r}_x \subset \gamma_x$ or $\mathbf{r}_x \supset \gamma_x$ then $\xi(x) = x$, and
- otherwise, letting d_x be the subinterval of a bounded by x and $\xi(x)$, d_x is the largest subinterval of a containing x and satisfying $\mathbf{r}_x \cap \gamma_y \neq \emptyset$ for all $y \in d_x$.

In addition, if $\mathbf{r}_x \subset \gamma_x$ or $\mathbf{r}_x \supset \gamma_x$, we set $d_x = \{x\}$. Moreover, for each $x \in a$, we have either,

- (I) γ_y with $y \in d_x$ contains the point $\mathbf{r}_x(0) = \beta_b \circ \Phi(x)$ only at $y = \xi(x)$ (Figure 9), or
- (II) $\xi(x)$ is an endpoint of a (Figure 10).

Then, since \mathbf{r}_y and γ_y are geodesic rays in $D \cong \mathbb{H}^2$ and $\gamma_y(0) = O$ for all $y \in a$, we have

Lemma 6.37.

- (i) $\text{length}(d_x) < \pi$.
- (ii) γ embeds $d_x \times (0, 1]$ into D for each $x \in a$.

Suppose that x is of Type (I). Then \mathbf{r}_x is contained in $\gamma(d_x \times [0, 1])$. Thus \mathbf{r}_x factors through via a unique embedding of the domain of \mathbf{r}_x into $d_x \times [0, 1] \subset \hat{F}$.

Suppose that x is of Type (II). We assume that $\xi(x) = 0$, since the case of $\xi(x) = \text{length}(\alpha)$ is similar. Let w be the maximal number in $a \cong [0, \text{length}(\alpha)]$ such that $\gamma(0) \cap \mathbf{r}_x \neq \emptyset$ for all $x \in [0, w]$. Then we have either

- (II-i) $\mathbf{r}_w(0)$ is on \mathbf{r}_0 (Figure 11) or
- (II-ii) \mathbf{r}_w contains O (Figure 12).

Suppose (II-i) holds. Then $x \in a$ satisfies $\xi(x) = 0$ if and only if $x \in [0, w]$. In addition, for no point x in $[0, w]$, we have $\gamma_x \subset \mathbf{r}_x$ or $\gamma_x \supset \mathbf{r}_x$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\angle(\gamma_y, \mathbf{r}_y) < \delta$ for all $y \in a$ (thus in particular at $y = 0$) and the angle between \mathbf{r}_w and γ_0 at $\beta_b \circ \Phi(w)$ is at most $\pi/2 + \delta$. Therefore, since \mathbf{r}_y does *not* intersect O for all $y \in [0, w]$, we see that $w < \pi/2 + \delta$ with sufficiently small $\delta > 0$ and in particular $w < \pi$. Let P the triangle bounded by $\beta_b(\Phi([0, w]))$, \mathbf{r}_0 , γ_0 ; then P is contained in E_0 . Since

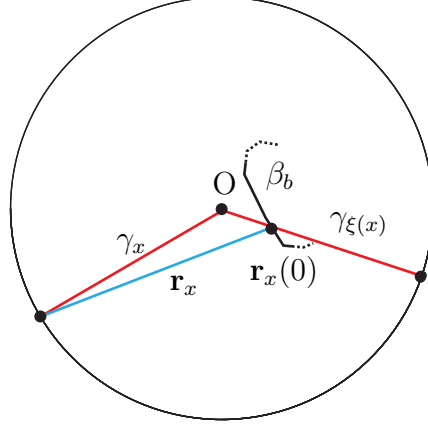


FIGURE 9.

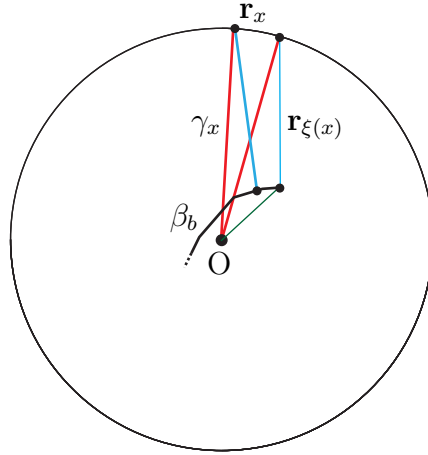


FIGURE 10.

$w < \pi$, γ embeds $[0, w] \times (0, 1]$ into D . Then $\gamma([0, w] \times [0, 1])$ and P have disjoint interiors and their union is topologically a disk. Thus, for each $x \in [0, w]$, the geodesic segment \mathbf{r}_x is naturally embedded in this disk, and we can naturally decompose the domain of \mathbf{r}_x into two segments and embed them into $[0, w] \times [0, 1] \subset \hat{F}$ and E_0 , so that \mathbf{r}_x factors through this embedding via γ .

Next suppose (II-ii) holds. Then $\xi(x) = 0$ for $x \in a$ if and only if $x \in [0, w]$. The geodesic in $D \cong \mathbb{H}^2$ containing γ_0 divides $\partial D \cong \mathbb{S}^1$ into

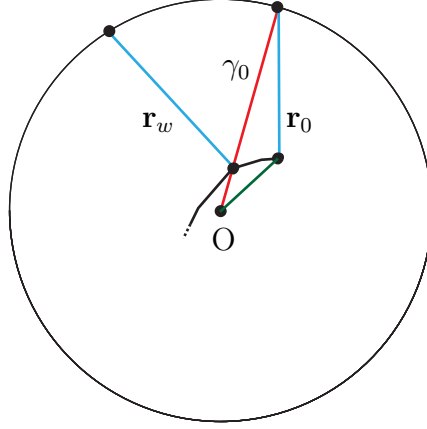


FIGURE 11.

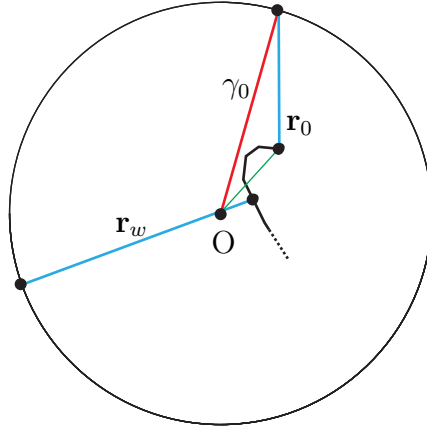


FIGURE 12.

two semicircles. Then $f_I([0, w])$ is strictly contained in one of those semicircles, and therefore we have $w < \pi$. Then γ_0 and $\beta_b \circ \Phi([0, w])$ are disjoint, and hence there is a disk in D bounded by γ_0 , \mathbf{r}_0 , q_w , and $\beta_b \circ \Phi([0, w])$, which we denote by P . This disk P is immersed into D but it may *not* be embedded in D . Nonetheless this immersion is at most two-to-one, and the image, in D , of such non-injective points is contained in a small neighborhood of the curve $\beta_b \circ \Phi([0, w])$. Let Q_0 be $[0, w] \times [0, 1] \subset Q$. Then we can decompose P into two simply

connected regions, by cutting along q_z with some appropriate $z \in [0, 1]$, and then isometrically embeds those regions into the disjoint union of E_0 and Q_0 . Therefore $\text{Area}(P) < \text{Area}(E_0) + \text{Area}(Q_0)$. Let \hat{F}_0 be the γ -image of $[0, w] \times [0, 1]$. Then \hat{F}_0 and P are topologically disks and their boundary circles share γ_0 . Thus the union of \hat{F}_0 and P along γ_0 is again a disk immersed into D . Then \mathbf{r} naturally embeds $[0, w] \times (0, 1]$ (essentially) onto this disk $\hat{F}_0 \cup_{\gamma_0} P$.

Therefore, letting F_0 denote $[0, w] \times [0, 1] \subset \text{Domain}(\gamma)$, we have

$$(5) \quad \text{Area}(F_0) < \text{Area}(\hat{F}_0) + \text{Area}(E_0) + \text{Area}(Q_0).$$

Then, for each $x \in [0, w]$, we can naturally decompose the domain of \mathbf{r}_x into at most three segments and embed them into $[0, w] \times [0, 1] \subset \text{Domain}(\gamma)$ and E_0 and $[0, w] \times [0, 1] \subset \text{Domain}(q)$, so that \mathbf{r}_x factors through this embedding via γ and q .

For all $x \in a$, let ι_x be the embedding of $[0, 1]$, the domain of \mathbf{r}_x , minus at most two points into $\hat{F} \sqcup E_0 \sqcup E_1 \sqcup Q$ defined as above for all types (I), (II-i), and (II-ii). Those points deleted from $[0, 1]$ change continuously in $x \in a$. Let X be the set of the deleted points $(x, t) \in a \times [0, 1]$ such that ι_x is noncontinuous at t . Then X has measured zero in F and it decomposes F into finitely many regions. Recalling that $F = a \times [0, 1]$, define $\iota: F \setminus X \rightarrow \hat{F} \sqcup E_0 \sqcup E_1 \sqcup Q$ by $\iota|_{\{x\} \times [0, 1]} = \iota_x$, which is continuous on $\{x\} \times [0, 1]$ minus at most two points, for each $x \in a$. Then ι is an isometric immersion (on each component of $F \setminus X$). Although ι is *not* necessarily an embedding (even up to measure zero as we see below), we have

Lemma 6.38. *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, for different $x, y \in a$, if $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap \hat{F} \neq \emptyset$, then $\pi - \delta < |x - y| < 2\pi$ and the \mathbf{r} -image of $\text{Im}(\iota_x) \cap \text{Im}(\iota_y)$ is contained in the δ -neighborhood of O .*

Proof. For $x, y \in a$, we have seen that $\text{Im}(\iota_x) \cap \hat{F} \subset d_x \times [0, 1]$ and $\text{Im}(\iota_y) \cap \hat{F} \subset d_y \times [0, 1]$. Assume $\text{Im}(\iota_x) \cap \text{Im}(\iota_y) \neq \emptyset$. Then $d_x \cap d_y \neq \emptyset$. By Lemma 6.37 (i), we must have $|x - y| < 2\pi$. By Lemma 6.31, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then \mathbf{r} embeds $d \times (0, 1]$ into D for all subintervals d of a with length less than $\pi - \delta$. Thus we have $\pi - \delta < |x - y| < 2\pi$ with sufficiently small $\delta > 0$.

Since γ is a fan, there is a natural quotient map from $\hat{F} \cong a \times [0, 1]$ to the quotient $a \times [0, 1]/a \times \{0\}$, which collapses $a \times \{0\}$ to a single point. Thus, for different x and y in a , $\{x\} \times [0, 1]$ and $\{y\} \times [0, 1]$ intersect only at $\{x\} \times \{0\} = \{y\} \times \{0\}$ in this quotient space $a \times [0, 1]/a \times \{0\}$. If $\epsilon > 0$ is sufficiently small, then ι_x and ι_y are δ -close to $\gamma(x)$ and $\gamma(y)$,

respectively, which correspond to $\{x\} \times [0, 1]$ and $\{y\} \times [0, 1]$. Thus, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\iota_x \cap \iota_y$ is contained in $a \times [0, \delta] \subset \hat{F}$ for all different $x, y \in a$. \square

Consider all points x on a such that the \mathbf{r}_x -image of $(0, 1]$ intersects the center O . There are only finitely many such points x , since the restriction of \mathbf{r} to $a \times (0, 1]$ is an immersion. Set $0 = x_0 < x_1 < \dots < x_{n+1} = \text{length}(\alpha)$ to be those points x together with the endpoints of a . For each $k = 0, 1, \dots, n$, set $F_k = [x_k, x_{k+1}] \times [0, 1] \subset F$. The center O divides \mathbf{r}_{x_k} into γ_{x_k} and q_{x_k} . Then, accordingly, let $\hat{F}_k = [x_k, x_{k+1}] \times [0, 1] \subset \hat{F}$ and $Q_k = [\Phi(x_k), \Phi(x_{k+1})] \times [0, 1] \subset Q$ for each $k = 0, 1, \dots, n$.

Lemma 6.39. *Assume that $\iota_x \cap \iota_y \cap \hat{F} \neq \emptyset$ for some $x, y \in a$. Then*

- *Both x, y are contained in the same interval $[x_k, x_{k+1}]$ for some $k \in \{0, 1, \dots, n\}$.*
- *All points in $[x_k, x_{k+1}]$ are of Type (I).*
- *There exist p, p' with $x_k < p < p' < x_{k+1}$ such that $\iota_x \cap \iota_y \cap \hat{F} \neq \emptyset$ for different $x, y \in [x_k, x_{k+1}]$ with $x < y$ if and only if $x \in [x_k, p]$ and $y \in [p', x_{k+1}]$.*

Proof. Suppose that $\iota_x \cap \iota_y \neq \emptyset$ for some $x, y \in a$ with $x < y$. By Lemma 6.38, $\pi - \delta < |x - y| < 2\pi$ with sufficiently small $\delta > 0$. Since $\mu_I \circ \Phi([x, y])$ and $\text{length}_h([x, y])$ are δ -close, we can assume that $\pi - \delta < \mu_I \circ \Phi([x, y]) < 2\pi + \delta$ by taking sufficiently small $\epsilon > 0$. Since μ_I corresponds to bending angles of β_b , the restriction of $\beta_b: b \rightarrow D$ to $\Phi([x_k, x_{k+1}])$ has at most a single self intersection. Therefore, ι_x and ι_y must intersect as in Figure 13 or 14. Then, in particular, for every $z \in (x, y)$, ι_z must be disjoint from O . Thus x and y are contained in the same interval $[x_k, x_{k+1}]$ for some $k \in \{0, 1, \dots, n\}$. Let Z be the set of pairs (x, y) of points in $[x_k, x_{k+1}]$ such that $x < y$ and $\iota_x \cap \iota_y \neq \emptyset$. From Figure 13 or 14, we see that $Z = [x_k, p] \times [p', x_{k+1}]$ for some p, p' with $x_k < p < p' < x_{k+1}$. Then x_k and x_{k+1} must be of Type (I), and therefore all points of $[x_k, x_{k+1}]$ are of Type (I). \square

We have seen that Type (II) occurs exactly when x is in a (possibly empty) interval containing an endpoint of x . Since x_1, x_2, \dots, x_n are of Type (I), this interval is contained in $[x_0, x_1]$ or $[x_n, x_{n+1}]$. Thus every $x \in [x_1, x_n]$ is of Type (I).

Claim 6.40. *Assume that all x in $[x_k, x_{k+1}]$ are of Type (I) for some $k \in \{0, 1, \dots, n\}$. Then $\text{Area}(F_k) \leq \text{Area}(\hat{F}_k) + \text{Area}(Q_k)$.*

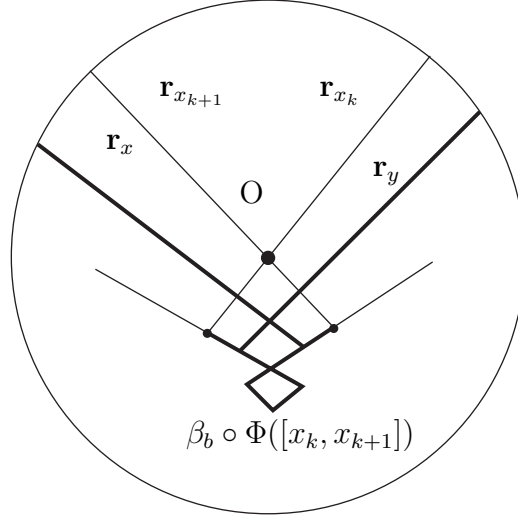


FIGURE 13.

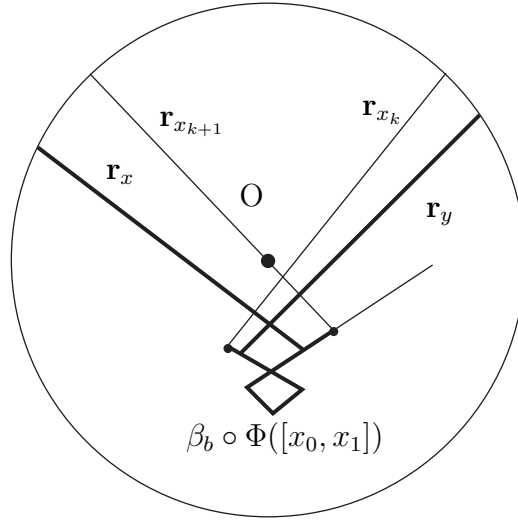


FIGURE 14.

Proof. Since $\gamma_{x_k} \subset \mathbf{r}_{x_k}$ and $\gamma_{x_{k+1}} \subset \mathbf{r}_{x_{k+1}}$, we see that $\iota(F_k)$ is contained in \hat{F}_k . Then, if $\iota|_{\hat{F}_k}$ is injective, the claim clearly holds. Thus, by Lemma 6.39, we can assume that there exists p, p' with $x_k < p < p' < x_{k+1}$ such that $\iota_x \cap \iota_y \cap \hat{F} \neq \emptyset$ for different $x, y \in [x_k, x_{k+1}]$ with $x < y$ if and only if $x \in [x_k, p]$ and $y \in [p', x_{k+1}]$ as in Figure 13 (if $k \in \{1, 2, \dots, n-1\}$) or Figure 14 (if $k = 0, n$). Thus $\iota(F_k) = \hat{F}_k$. Suppose, in addition, $k \in \{1, 2, \dots, n-1\}$. Then consider the disk bounded by $\beta_b \circ \Phi([x_k, x_{k+1}])$, \mathbf{r}_{x_k} , and $\mathbf{r}_{x_{k+1}}$, which we denote

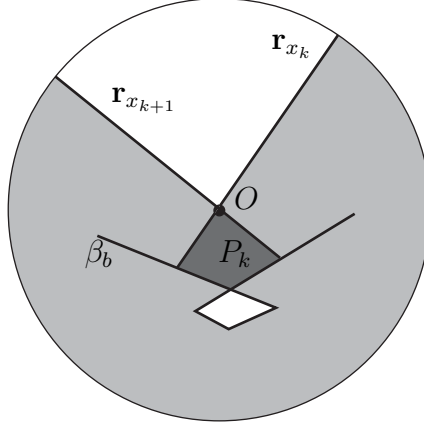


FIGURE 15.

by P_k (see Figure 15). Then, since $q_{x_k} \subset \mathbf{r}_{x_k}$ and $q_{x_k} \subset \mathbf{r}_{x_k}$, we see that P_k is contained in $q(Q_k)$. Thus $\text{Area}(P_k) \leq \text{Area}(Q_k)$. We see that \mathbf{r} is injective on $F_k \setminus \mathbf{r}^{-1}(P_k)$. In addition $\mathbf{r}^{-1}(P_k) \cap F_k$ has exactly two connected components, and \mathbf{r} isometrically embeds each component onto P_k . Therefore we have

$$\text{Area}(F_k) = \text{Area}(\iota(F_k)) + \text{Area}(P_k) \leq \text{Area}(\hat{F}_k) + \text{Area}(Q_k)$$

as desired.

The proof is similar when $k = 0$ or $k = n$. □

Claim 6.41. *Suppose that $[x_0, x_1]$ contains a subinterval of Type (II-i). Then ι embeds $F_0 \setminus X$ into $\hat{F}_0 \sqcup E_0$; therefore $\text{Area}(F_0) \leq \text{Area}(\hat{F}_0) + \text{Area}(E_0)$. Similarly, suppose that $[x_n, x_{n+1}]$ contains a subinterval of Type (II-i). Then ι embeds $F_n \setminus X$ into $\hat{F}_n \sqcup E_n$; therefore $\text{Area}(F_n) \leq \text{Area}(\hat{F}_n) + \text{Area}(E_n)$.*

Proof. Let $[0, w] \subset [0, \text{length}(\alpha)]$ be the (maximal) subinterval of Type (II -i), if there exists. Then $[0, w]$ is contained in $[0, x_1]$. By Lemma 6.39, ι embeds $F_0 \cap \iota^{-1}(\hat{F})$ into \hat{F}_0 . In addition we have seen that ι embeds $[0, w] \times [0, 1]$ into the disjoint union of $[0, w] \times [0, 1] \subset \text{Domain}(\gamma)$ and E_0 . Thus ι embeds $F_0 \setminus X$ into $\hat{F}_0 \sqcup E_0$.

We can similarly prove the analogous claim about F_n . □

In addition, by (5), we have an analogous statement for Type (II-ii):

Claim 6.42. *Suppose that $[x_0, x_1]$ contains a subinterval of Type (II-ii). Then ι embeds $F_0 \setminus X$ into $\hat{F}_0 \sqcup E_0 \sqcup Q_0$; therefore $\text{Area}(F_0) \leq \text{Area}(\hat{F}_0) + \text{Area}(E_0) + \text{Area}(Q_0)$. Similarly suppose that $[x_n, x_{n+1}]$ contains a subinterval of Type (II-ii). Then ι embeds $F_n \setminus X$ into $\hat{F}_n \sqcup E_1 \sqcup Q_n$; therefore $\text{Area}(F_n) \leq \text{Area}(\hat{F}_n) + \text{Area}(E_1) + \text{Area}(Q_n)$.*

By Claim 6.40, Claim 6.41 and Claim 6.42, we have

$$\begin{aligned} \text{Area}(F) &= \sum_{k=0}^n \text{Area}(F_k) \\ &\leq \sum_{k=0}^n \text{Area}(\hat{F}_k) + \sum_{k=0}^n \text{Area}(Q_k) + \text{Area}(E_0) + \text{Area}(E_1) \\ &= \text{Area}(\hat{F}) + \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1). \end{aligned}$$

6.33

6.32

6.27

6.9. The proof of Theorem 6.1. Recall that there are three cases in the theorem.

6.9.1. *Case (II).* This case is equivalent to the following claim:

Proposition 6.43. *Let $C = (\tau, L)$ and $C' = (\tau', L')$ be different projective structures with the same holonomy $\rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$. Assume $[L] = [L'] \in \mathcal{PML}$ and $L \leq L'$. Then $\text{Gr}_{L'-L}(C) = C'$.*

Proof of Claim 6.43. Let $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ and $\beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3$ denote the bending maps associated with $C = (\tau, L)$ and $C' = (\tau', L')$, respectively. Since $[L] = [L']$, by Proposition 5.1, we have $\beta = \beta'$ and, in particular, $\tau = \tau'$. Then

Lemma 6.44. *L and L' have only periodic leaves.*

Proof. Assume that the projective measured lamination $[L] = [L']$ contains a minimal irrational lamination. Then, this irrational lamination supports sublaminations N and N' of L and L' , respectively. Set $L = (\lambda, \mu)$ and $L' = (\lambda', \mu')$, where λ and λ' are geodesic laminations on τ and μ and μ' are transversal measures. Recall that the transversal measures μ and μ' correspond to (infinitesimal) exterior angles of the bending maps β and β' , respectively ([12, 16]). Then, since $\beta = \beta'$ and the lamination N and N' are irrational, we must have $N = N'$. Therefore, since $[L] = [L'] \in \mathcal{PML}$ and the laminations N and N' are sublaminations of L and L' , we have $L = L'$. Since $\tau = \tau'$, this is a contradiction to the assumption that $C \neq C'$. \square

By this lemma, λ and λ' are the same multiloops on $\tau = \tau'$, which is the support of $L' - L$. Then, since $\beta = \beta'$, the weight of each loop of $L' - L$ must be a multiple of 2π . This completes the proof. \square

6.9.2. *Case (III).* Suppose that $L = \emptyset$ in $\mathcal{ML}(S)$. Then clearly ρ is fuchsian. Given a projective structures $C' = (\tau', L')$ with the same holonomy ρ , letting $L' = (\lambda', \mu')$, we can regard $L = \emptyset$ as $0 \cdot L' = (\lambda', 0)$. In this way, we can regard L and L' as having the same supporting lamination. Therefore we can apply the proof of Case (II), and see that $C' = Gr_{L'}(C)$.

6.9.3. *Case (I).* Let $C = (f, \rho)$ be a projective structure on S with holonomy ρ . Set $C = (\tau, L)$ to be Thurston's coordinates of C . Let U be the neighborhood of $[L]$ in $\mathcal{PM}\mathcal{L}(S)$, obtained by applying Proposition 6.10 to C . Then, for every $C' = (f', \rho) \in \mathcal{P}_\rho$ with $[L'] \in U$, there are a topological (fat) traintrack $T = \cup_j B_j$ ($j = 1, 2, \dots, n$) on S , where B_j are branches of T , and marking homeomorphisms $\phi: S \rightarrow C$ and $\phi': S \rightarrow C'$ that satisfy (I) and (II) in Proposition 6.10. Set $C' = (\tau', L')$ to be Thurston's coordinates of C' . Then we assume that $[L] \neq [L']$ since it is an assumption of Case (I).

Regard S as the union of $S \setminus T$ and $T = \cup_j B_j$. Then, this decomposition of S induces decompositions of C and of C' via ϕ and ϕ' , respectively:

$$\begin{aligned} C' &= (C' \setminus \phi'(T)) \cup \phi'(T) = (C' \setminus \phi'(T)) \cup (\cup_j \phi'(B_j)). \\ C &= (C \setminus \phi(T)) \cup \phi(T) = (C \setminus \phi(T)) \cup (\cup_j \phi(B_j)). \end{aligned}$$

Then Proposition 6.10 (I), $\phi' \circ \phi^{-1}$ yields an isomorphism from $(C \setminus \phi(T))$ to $(C' \setminus \phi'(T))$ compatible with the developing maps f and f' . In addition, by Proposition 6.10 (II - ii), for each $j = 1, 2, \dots, n$, we have $\phi'(B_j) = Gr_{M_j}(\phi(B_j))$ for some admissible multiarc M_j on $\phi(B_j)$. Then, by the definition of an admissible multiarc, each arc of M_j connects the outermost ties of $\phi(B_j)$.

Let $\kappa: C \rightarrow \tau$ and $\kappa': C' \rightarrow \tau'$ be the collapsing maps. Then by Proposition 6.10 (iii), κ and κ' descend the traintracks $\phi(T)$ on C and $\phi'(T)$ on C' to traintracks on τ and τ' carrying L and L' , respectively; setting $L = (\lambda, \mu)$ and $L' = (\lambda', \mu')$, indeed $\frac{1}{2\pi}[\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))]$ is a good approximation of the number of arcs of M_j , for each $j = 1, 2, \dots, n$.

Since $\kappa' \circ \phi'(T)$ carries L' , the n -tuple $\{\mu'(\kappa' \circ \phi'(B_j))\}_{j=1}^n \in \mathbb{R}_{>0}^n$ satisfies the switch conditions of the traintrack $\kappa' \circ \phi'(T)$ on τ' and similarly, since $\kappa \circ \phi(T)$ carries L , the n -tuple $\{\mu(\kappa \circ \phi(B_j))\}_{j=1}^n \in \mathbb{R}_{\geq 0}^n$ satisfies the switch conditions of the traintrack $\kappa \circ \phi(T)$ on τ . Since $\kappa' \circ \phi': S \rightarrow \tau$ and $\kappa \circ \phi: S \rightarrow \tau$ identify the traintrack T on S with

those image traintracks on τ and τ , respectively, the n -tuple of their differences $\{\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))\}_{j=1}^n \in \mathbb{R}_{\geq 0}^n$ satisfies the switch conditions as well. Therefore, by the approximation, the n -tuple of the numbers of the arcs of M_j ($j = 1, 2, \dots, n$) also satisfies the switch conditions. Thus the union $\cup_j M_j =: M$ is a multiloop carried by the traintrack $\phi(T) \subset C$, up to proper isotopies of M_j on $\phi(B_j)$ through admissible multiarcs.

For each $j = 1, 2, \dots, n$, $\phi'(B_j) = Gr_{M_j}(\phi(B_j))$ still holds after the isotopy of M_j (Lemma 6.5). Since $\phi(T)$ carries L and $\phi'(T)$ carries L' , we can regard $L' - L$ as a measured lamination on S carried by T whose weight on the branch B_j is $\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))$ for each j . Thus $L' - L$ is a good approximation of M .

We last compare $\phi(T) \subset C$ and $\phi'(T) \subset C'$ as projective structures on T . Let B_i and B_j be adjacent branches of T and let m_i and m_j be arcs of M_i and M_j , respectively, that share an endpoint, so that $m_i \cup m_j$ is a simple arc on $B_i \cup B_j$. Since B_j and B_i are supported on a round cylinder, the projective structure inserted by the grafting $Gr_{m_i \cup m_j}$ of $B_i \cup B_j$ is exactly the union of projective structures inserted by the graftings Gr_{m_i} of B_i and Gr_{m_j} of B_j . Since this happens for all adjacent arcs, we have

$$\phi'(T) = \cup_j \phi'(B_j) = \cup_j Gr_{M_j}(\phi(B_j)) = Gr_M(\phi(T))$$

(a similar argument is used in [2]). Hence

$$C' = (C' \setminus \phi'(T)) \cup \phi'(T) = (C \setminus \phi(T)) \cup [Gr_M(\phi(T))] = Gr_M(C).$$

6.1

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